Solutions/ Hints to homework 10

March 26, 2008

section 6.1

13b) Clearly $I \subset R$. We need to show that $R \subset I$. Let $u \in I$ be a unit and let $v \in R$ be such that $u \cdot v = 1$. Since $u \in I$ we get that $1 = u \cdot v \in I$. Therefore if $r \in R$ then $r = r \cdot 1 \in I$.

14. Suppose that $I \subset F$ is a non zero ideal, i.e. it contains a non zero element x. Since F is a field, we have that x is a unit. By the previous exercise we get that $I = F$.

15b. We use Thm. 6.1. First note that $\bigcap I_r$ is non empty, since 0 is in every I_r , hence $0 \in \bigcap I_r$.

Now, let s be an arbitrary element of the ring R and let $a, b \in \bigcap I_r$, i.e. $a, b \in I_r$ for every r. Since each I_r is an ideal we get that $a - b$ and sa are in each I_r, hence $a - b$ and $ra \in \bigcap I_r$.

16. see solution to homework 5, section 3.2 ex. 5b

43. a) and b) are straightforward.

c) Consider cosets

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + I,$$

where $(a, b) \in \mathbb{R} \times \mathbb{R}$. Note that if $(a, b) \neq (a', b') \in \mathbb{R} \times \mathbb{R}$ (i.e. $a - a' \neq 0$ or $b - b' \neq 0$) then

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + I \neq \begin{pmatrix} a' & 0 \\ 0 & c' \end{pmatrix} + I,$$

since

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} - \begin{pmatrix} a' & 0 \\ 0 & c' \end{pmatrix} = \begin{pmatrix} a - a' & 0 \\ 0 & c - c' \end{pmatrix} \notin I.$$

Therefore we constructed an infinite family of cosets parameterized by elements of $\mathbb{R} \times \mathbb{R}$.

section 6.2

7. Let $T = 3\mathbb{Z}, I = 6\mathbb{Z}$. I leave it for you to check that I is an ideal. Note that $T/I = \{0 + I, 3 + I\}$.

We have

$$(0 + I) + (0 + I) = 0 + I$$
$$(0 + I) + (3 + I) = 3 + I = (3 + I) + (0 + I)$$
$$(3 + I) + (3 + I) = 0 + I$$

and

$$(0 + I) \cdot (0 + I) = 0 + I$$
$$(0 + I) \cdot (3 + I) = 0 + I = (3 + I) \cdot (0 + I)$$
$$(3 + I) \cdot (3 + I) = 9 + I = 3 + I.$$

The coset $3 + I$ is the multiplicative identity in T/I. Also, T/I is a field, since the non trivial coset $3 + I$ has an inverse. In fact, $T/I \cong \mathbb{Z}_2$.

14. Let $a + I, b + i \in R/I$. We have: $(a + I)(b + I) = ab + I$. Since $ab - ba \in I$,we also have

$$(b + I)(a + I) = ba + I = ba + (ab - ba) + I = ab + I,$$

i.e. R/I is commutative.

31. Let's use the First Isomorphism theorem. Define a map

$$f : S \to \mathbb{R} \times \mathbb{R}$$

$$f(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = (a, c).$$

I leave it for you to check that f is a ring homomorphism (i.e that f preserves addition and multiplication).

We show now that f is surjective: if $(a, b) \in \mathbb{R} \times \mathbb{R}$ then

$$f(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) = (a, b).$$

Next we show that $kerf = I$. Clearly $I \subset kerf$, because if $x \in I$ then $f(x) = (0, 0)$. Next we check that $kerf \subset I$: Let

$$x = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in kerf.$$

That means that $(a, b) = f(x) = (0, 0)$, i.e. $a = 0$ and $b = 0$. Hence

$$x = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in I.$$

Now we apply the First Isomorphism theorem and we get that $S/I \cong \mathbb{R} \times \mathbb{R}$.