

Political Science 239  
Regression Discontinuity

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# 1 Regression Discontinuity

The regression discontinuity design is a quasi-experimental design with the fundamental characteristic that the probability of receiving treatment changes discontinuously as a function of one or more underlying variables.

As we have mentioned a thousand times by now, we are usually interested in the effect of some binary treatment  $T_i$  on an outcome  $y_i$  and the fundamental problem of causal inference arises because we only observe the outcome under one of the two possible treatments. Let  $y_{i0}$  be the outcome under  $T_i = 0$  and  $y_{i1}$  be the outcome under  $T_i = 1$ . The observed outcome can be written as  $y_i = \alpha_i + T_i \cdot \beta_i$  where  $\alpha_i \equiv y_{i0}$  and  $\beta_i \equiv y_{i1} - y_{i0}$ . The discontinuity design can take one of two forms, the sharp design and the fuzzy design. With a sharp design,  $T_i$  is known to depend in a deterministic way on some observable variable  $z_i$ ,  $T_i = f(z_i)$ , where  $z_i$  takes a on a continuum of values. The function  $T = f(z)$  is discontinuous at the point  $z_0$ , and this point is assumed to be known. With a fuzzy design,  $T_i$  is a random variable given  $z_i$ , but the conditional probability

$$\begin{aligned} f(z) \equiv E[T_i | z_i = z] &= \\ &= 1 \cdot \Pr[T_i = 1 | z_i = z] + 0 \cdot \Pr[T_i = 0 | z_i = z] \\ &= \Pr[T_i = 1 | z_i = z] \end{aligned}$$

is discontinuous at  $z_0$ . The fuzzy design differs from the sharp design in that the treatment is not a deterministic function of  $z_i$ , there are other variables that determine assignment to treatment. The fuzzy design is similar to the sharp design in that the probability of receiving treatment viewed as a function of  $z_i$ ,  $\Pr[T_i = 1 | z_i]$ , is discontinuous at  $z_0$ . This is summarized in the following assumption:

**Assumption 1** (i) The limits  $T^+ \equiv \lim_{z \rightarrow z_0^+} E[T_i | z_i = z]$  and  $T^- \equiv \lim_{z \rightarrow z_0^-} E[T_i | z_i = z]$

exist. (ii)  $T^+ \neq T^-$ .

Note that what this assumption means is that the probability of receiving treatment jumps discontinuously at  $z_0$ .

## 1.1 Constant Treatment Effects

Suppose that the treatment effect  $\beta$  is constant across all individuals and let  $\epsilon$  denote an arbitrary small number. Suppose further that we have reason to believe that in the absence of treatment, persons close to the threshold  $z_0$  are similar. If this is true, we would expect

$$E[\alpha_i | z_i = z_0 + \epsilon] \cong E[\alpha_i | z_i = z_0 - \epsilon]$$

This motivates the second assumption:

**Assumption 2** :  $E[\alpha_i | z_i = z_0]$  is continuous in  $z$  at  $z_0$ .

We can easily prove that under this continuity restriction  $\beta$  is nonparametrically identified.

**Theorem 1** : Suppose that  $\beta_i = \beta$  for all  $i$  and that assumptions 1 and 2 hold. Then

$$\beta = \frac{y^+ - y^-}{T^+ - T^-} \tag{1}$$

where  $y^+ \equiv \lim_{z \rightarrow z_0^+} E[y_i | z_i = z]$  and  $y^- \equiv \lim_{z \rightarrow z_0^-} E[y_i | z_i = z]$

**Proof** : The mean difference in outcomes for persons above and below the discontinuity point is

$$E[y_i | z_i = z_0 + \epsilon] - E[y_i | z_i = z_0 - \epsilon] =$$

$$\begin{aligned}
&= E[\alpha_i + T_i \cdot \beta_i \mid z_i = z_0 + \epsilon] - E[\alpha_i + T_i \cdot \beta_i \mid z_i = z_0 - \epsilon] \\
&= E[\alpha_i \mid z_i = z_0 + \epsilon] - E[\alpha_i \mid z_i = z_0 - \epsilon] \\
&+ \beta (\cdot E[T_i \mid z_i = z_0 + \epsilon] - E[T_i \mid z_i = z_0 - \epsilon])
\end{aligned}$$

And by Assumption 2 we have

$$\lim_{z \rightarrow z_0^+} E[y_i \mid z_i = z] - \lim_{z \rightarrow z_0^-} E[y_i \mid z_i = z] = \beta \left\{ \lim_{z \rightarrow z_0^+} E[T_i \mid z_i = z] - \lim_{z \rightarrow z_0^-} E[T_i \mid z_i = z] \right\}$$

because since  $E[\alpha_i \mid z_i = z_0]$  is continuous in  $z$  at  $z_0$  the left limit and the right limit are equal. The conclusion then follows:

$$\beta = \frac{\lim_{z \rightarrow z_0^+} E[y_i \mid z_i = z] - \lim_{z \rightarrow z_0^-} E[y_i \mid z_i = z]}{\lim_{z \rightarrow z_0^+} E[T_i \mid z_i = z] - \lim_{z \rightarrow z_0^-} E[T_i \mid z_i = z]} = \frac{y^+ - y^-}{T^+ - T^-}$$

and Assumption 1 guarantees that  $T^+ - T^-$  is non-zero.

The sharp design is a special case of the fuzzy design with  $T^+ = 1$  and  $T^- = 0$ . Therefore, in the special case of sharp regression discontinuity  $\beta$  is identified by

$$\beta = y^+ - y^- \tag{2}$$

## 1.2 Heterogeneous Treatment Effects

Now consider the identification problem when the treatment effect is variable across individuals. To generalize the identification strategy we make the following assumption:

**Assumption 3** :  $E[\beta_i \mid z_i = z]$  regarded as a function of  $z$  is continuous at  $z_0$ .

The following theorem proves that the average treatment effect at  $z_0$ ,  $E[\beta_i \mid z_i = z]$ , is nonparametrically identified under assumptions 1, 2 and 3 and a weak form of conditional independence.

**Theorem 2** : Suppose that  $T_i$  is independent of  $\beta_i$  conditional on  $z_i$  near  $z_0$ . Suppose further that assumptions 1, 2 and 3 hold. Then

$$E[\beta_i | z_i = z_0] = \frac{y^+ - y^-}{T^+ - T^-} \quad (3)$$

**Proof** :The mean difference in outcomes for persons above and below the discontinuity point is

$$\begin{aligned} & E[y_i | z_i = z_0 + \epsilon] - E[y_i | z_i = z_0 - \epsilon] = \\ &= E[\alpha_i + T_i \cdot \beta_i | z_i = z_0 + \epsilon] - E[\alpha_i + T_i \cdot \beta_i | z_i = z_0 - \epsilon] \\ &= E[\alpha_i | z_i = z_0 + \epsilon] - E[\alpha_i | z_i = z_0 - \epsilon] \\ &+ E[\beta_i \cdot T_i | z_i = z_0 + \epsilon] - E[\beta_i \cdot T_i | z_i = z_0 - \epsilon] \end{aligned}$$

By conditional mean independence we have

$$E[\beta_i \cdot T_i | z_i = z_0 \pm \epsilon] = E[\beta_i | z_i = z_0 \pm \epsilon] \cdot E[T_i | z_i = z_0 \pm \epsilon]$$

which yields

$$\begin{aligned} & E[y_i | z_i = z_0 + \epsilon] - E[y_i | z_i = z_0 - \epsilon] = \\ &= E[\alpha_i | z_i = z_0 + \epsilon] - E[\alpha_i | z_i = z_0 - \epsilon] + \\ & \quad \{E[\beta_i | z_i = z_0 + \epsilon] \cdot E[T_i | z_i = z_0 + \epsilon]\} \cdot \{E[\beta_i | z_i = z_0 - \epsilon] \cdot E[T_i | z_i = z_0 - \epsilon]\} \end{aligned}$$

and combining this with assumptions 2 and 3 we obtain

$$\lim_{z \rightarrow z_0^+} E[y_i | z_i = z] - \lim_{z \rightarrow z_0^-} E[y_i | z_i = z] = \quad (4)$$

$$E[\beta_i | z_i = z_0] \left\{ \lim_{z \rightarrow z_0^+} E[T_i | z_i = z] - \lim_{z \rightarrow z_0^-} E[T_i | z_i = z] \right\} \quad (5)$$

and it follows that

$$E[\beta_i | z_i = z_0] = \frac{\lim_{z \rightarrow z_0^+} E[y_i | z_i = z] - \lim_{z \rightarrow z_0^-} E[y_i | z_i = z]}{\lim_{z \rightarrow z_0^+} E[T_i | z_i = z] - \lim_{z \rightarrow z_0^-} E[T_i | z_i = z]} = \frac{y^+ - y^-}{T^+ - T^-}$$

Again, with a sharp design this expression simplifies to:

$$E[\beta_i | z_i = z_0] = y^+ - y^-$$

The assumption of conditional independence is strong, as we already know: it means that individuals do not select into treatment on the basis of the benefits that they expect to get from treatment. We can define an alternative set of conditions that does allow for selection into treatment on the basis of anticipated gains. The assumption that will be used instead of assumption 3 is the following.

**Assumption 4** : (i)  $(\beta_i, T_i(z))$  is jointly independent of  $z_i$  near  $z_0$ . (ii) There exists  $\delta > 0$  such that  $T_i(z_0 + \epsilon) \geq T_i(z_0 - \epsilon)$  for all  $0 < \epsilon < \delta$ .

The following theorem shows that we can identify  $\beta$  under assumptions 1,2 and 4.

**Theorem 3** : Suppose assumptions 1,2 and 4 hold. We then have

$$\lim_{\epsilon \rightarrow 0} E[\beta_i | T_i(z_0 + \epsilon) - T_i(z_0 - \epsilon) = 1] = \frac{y^+ - y^-}{T^+ - T^-}$$

Note that for  $\epsilon$  sufficiently small, the event  $T_i(z_0 + \epsilon) - T_i(z_0 - \epsilon) = 1$  on which we are conditioning corresponds to the subgroup of people for whom treatment changes discontinuously at  $z_0$ . So this expression identifies the *local* average treatment effect at  $z_0$ .

You probably will have noticed the oddity of this argument. The identification of the treatment effect is possible because we compare persons arbitrarily close to the discontinuity point  $z_0$  who did and did not receive treatment. Unless treatment effects are homogeneous across all individuals, treatment effects can *only* be identified at  $z = z_0$ . This is a limitation of RD: it only identifies the treatment effect locally at the point at which the probability of receiving treatment changes discontinuously.

### 1.3 Estimation

Since the ratio  $\frac{y^+ - y^-}{T^+ - T^-}$  identifies the treatment effect at  $z_0 = z$ , we can estimate the treatment effect by

$$\frac{\hat{y}^+ - \hat{y}^-}{\hat{T}^+ - \hat{T}^-}$$

So we want to estimate the limits  $y^+$ ,  $y^-$ ,  $T^+$  and  $T^-$ . These limits are nothing but conditional expectations, so we can estimate these limits using weighted averages. The strategy is as follows. To simplify, consider the sharp design so that we only need to estimate  $y^+$  and  $y^-$ . In order to estimate  $y^+$ , we use the observed  $y_i$  to the right of  $z_0$  to estimate  $\hat{E}[y | z > z_0]$  and the observed  $y_i$  to the left of  $z_0$  to estimate  $\hat{E}[y | z < z_0]$ . The important point is that we will weight these observations according to how far away they are from the discontinuity point. The further away the observations are from the discontinuity point  $z_0$ , the less weight they will receive in the average. Observations that are too far will receive a weight of almost zero.

### 1.4 Discussion

In sum, regression discontinuity designs involve a dichotomous treatment that is deterministic function of a single, observed, continuous covariate (the  $z$  we used above). Many times people refer to this continuous variable as the "score". The essential assumption is that the average outcome for individuals that are just below the discontinuity point must be

a valid counterfactual for the treated group just above the discontinuity point. This is,  $E[y_0 | z = z_0]$  and  $E[y_1 | z = z_0]$  must be continuous in  $z$  at  $z_0$ .

The point made by Lee (2008) is that it is hard to determine the plausibility of these assumptions because they don't refer to the treatment assigning process. Rather, they refer to a mathematical condition that is untestable. Once again, this is because potential outcomes are not observed. Suppose treatment is assigned once certain score  $z$  is above  $z_0$ . Lee's contribution is to show that we can apply an RD design even when people have the ability to influence the score, as long as the score is also influenced by random chance. The key issue is that conditional on individuals choices and characteristics, the probability density of the score must be continuous. So in this setup each individual can take actions that influence their probability of treatment, but given the random component of the score, each individual's probability of receiving treatment is *between* zero and one. This means that while individuals can, say, increase their probability of receiving treatment, they cannot guarantee that they will in fact receive treatment. Lee's point is important because it shows that even when we have non-random selection into treatment we can identify impact estimates that share the same validity as those from a randomized experiment. Localized random treatment assignment can occur even in the presence of endogenous sorting, as long as agents don't have the ability to sort *precisely* around the threshold. If they can, the density of the score is likely to be discontinuous. The key assumption is the continuity of the density of the score for every individual. When this condition holds, we have random assignment of treatment around the threshold. Hence, we have local independence. And this has testable implications, mainly that all pre-treatment characteristics should be indistinguishable around the threshold.

The important point is that around the threshold it is *as if* we had random assignment and therefore we can verify whether all characteristics determined pre-treatment are actually similar between treatments and controls.

## References

- Hahn J., P. Todd, and W.V. der Klaauw, 2001. Identification and Estimation of Treatment Effects with a Regression-Discontinuity Design. *Econometrica*.
- Lee, D., 2008. Randomized experiments from non-random selection in U.S. House election. *Journal of Econometrics*, 142(2), pp. 675-697.
- **If you want to know more about RD, check out the Volume 142, Issue 2 of the Journal of Econometrics. This issue was entirely devoted to RD.**