NOTES ON TAYLOR SERIES

The Taylor series for a function $f(x)$ centered at c is a power series centered at c which equals $f(x)$ when it converges. We can work out the coefficients for this power series:

Taylor Series Formula: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

Using the definition to compute a Taylor series can be tedious because you have to compute lots of derivatives of $f(x)$ until you see a pattern. So we want some shortcuts for computing Taylor series.

1. Substitution & Algebra

Our first trick is to use Taylor series we already know and a little algebra to compute new Taylor series.

ex 1. Compute the Taylor series of $x\cos(x^3)$ centered at $x=0$.

Soln. We already know the Taylor series for $\cos(x)$ centered at $x=0$, so let's use that.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$ $\longrightarrow$ let's first plug in x^3, and then multiply by x.

$$\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

So...

$$x\cos(x^3) = x\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!}.$$ We did it!

ex 2. Find the Taylor series of $\ln(8-x^3)$ centered at 0 and its radius of convergence.

Soln We worked out the Taylor series for $\ln(1-x)$ in class, which looks similar to this function. Let's try to do some algebra to get from $\ln(1-x)$ to $\ln(8-x^3)$.

$$\ln(8-x^3) = \ln\left(8\left(1-\frac{x^3}{8}\right)\right) = \ln(8) + \ln\left(1-\frac{x^3}{8}\right)$$

$\ln(a \cdot b) = \ln(a) + \ln(b)$

Aha! We can substitute $x^3/8$ for x in $\ln(1-x)$ to get this series.

Okay, now recall that

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

w/ convergence for $|x| < 1$.

So... $\ln(8-x^3) = \ln(8) + \ln(1-x^3/8)$

$$= \ln(8) - \sum_{n=1}^{\infty} \frac{(x^3/8)^n}{n}$$

$$= \ln(8) - \sum_{n=1}^{\infty} \frac{x^{3n}}{n\,8^n}$$

$\therefore$ $\ln(8-x^3) = \ln(8) - \sum_{n=1}^{\infty} \frac{x^{3n}}{n\,8^n}$ and this converges for $\left|\frac{x^3}{8}\right| < 1$

$\Rightarrow |x|^3 < 8$

$\Rightarrow |x| < 2$.

this is the constant term of the Taylor series.

So the radius of convergence is $R = 2$. How nice!

2. Differentiation & Integration

Our second trick is to take derivatives and integrals of know Taylor series to find new ones.

ex 1. Find the Taylor series centered at $x=0$ of $\frac{1}{(1+x^2)^2}$.
what is its radius of convergence?

Soln. This one is tricky! I first start with the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

Next I take its derivative to get:

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$

Taking derivatives or integrals doesn't change the radius of convergence, so it's still 1.

Now plug in $-x^2$ for x.

$$\frac{1}{(1+x^2)^2} = \sum_{n=0}^{\infty} n(-x^2)^{n-1} = \sum_{n=0}^{\infty} n(-1)^{n-1} x^{2n-2}$$

and this converges for $|-x^2|<1 \implies |x|<1$.
So the radius is 1.

ex 2. Find the first 3 nonzero terms of the Taylor series of $\arcsin(x)$ centered at $x=0$.

Soln. The trick for inverse trig functions is to first take their derivatives. $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$. Notice that $\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$

looks like a binomial series with $-x^2$ substituted for x and $p=-\frac{1}{2}$

Since $(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \ldots$

we get

$$(1-x^2)^{-\frac{1}{2}} = 1 + (-\tfrac{1}{2})(-x^2) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2}(-x^2)^2 + \ldots$$

$$(1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \ldots$$

And since we know that $\frac{d}{dx}\arcsin(x) = (1-x^2)^{-\frac{1}{2}}$, we can integrate to get

$$\arcsin(x) = c + x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \ldots$$

Plugging in $x=0$ let's us solve for c.

$$0 = \arcsin(0) = c.$$

Thus, $$\arcsin(x) = \underbrace{x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5}_{\text{first 3 nonzero terms.}} + \ldots$$

Those are our two main tricks. Next I'll go over some applications of Taylor series.

1. Computing Limits.

If you want to compute a limit like $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$, the first thing you try is plugging in $x=0$.

But there's a problem... you get $\frac{0}{0}$ which is undefined.

We can use Taylor series to compute such limits. Recall that

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$, if we subtract $1+x$ we get

$e^x - 1 - x = \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ and then dividing by x^2 gives

$\frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \frac{x}{6} + \ldots$ We can now take the limit as $x \to 0$ by plugging in!

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{1}{2} + \frac{x}{6} + \cdots = \frac{1}{2}.$$

This is the bad-ass alternative to L'hospital's Rule.

Try using this to compute $\lim_{x\to 0} \frac{\sin x}{x}$ and $\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{x^4}$.

2. Computing large derivatives

whenever you're asked to compute something like $f^{(100)}(0)$ for a given function $f(x)$, the trick is always to use Taylor series

ex 1. Say $f(x) = x^2 \sin(x)$. what's $f^{(11)}(0)$? what's $f^{(100)}(0)$?

Soln. Since they're asking for derivatives at 0, we want to consider the Taylor series for $f(x)$ centered at $x=0$. Recall that

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ is the Taylor series. So we can find $f^{(11)}(0)$ by looking at the coefficient of x^{11} in the Taylor series for $f(x)$. But wait! We have another way to compute this Taylor series. Start with the series for $\sin(x)$.

$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, multiply by x^2 to get

$$x^2 \sin(x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}$$

The Taylor series formula tells us that the coefficient of x^{11} in this series expansion of $x^2\sin(x)$ is $\frac{f^{(11)}(0)}{11!}$. Setting $2n+3 = 11$ we get $n=4$ corresponds to x^{11} in the expansion just found. Plugging in $n=4$ we then have

$$\frac{f^{(11)}(0)}{11!} = \frac{(-1)^4}{9!} \Longrightarrow f^{(11)}(0) = \frac{11!}{9!} = 11 \cdot 10 = 110.$$

So $f^{(11)}(0) = 110$.

To compute $f^{(100)}(0)$ there's a short cut: notice that $2n+3$ is always odd which means that the Taylor series for $x^2\sin(x)$ only has odd power terms. In other words, all the even power terms are 0. So $\frac{f^{(100)}(0)}{100!} = 0 \Rightarrow f^{(100)}(0) = 0$.

3. Estimating Integrals

Most integrals are impossible to compute exactly by hand. And yet, the world must keep turning — so we need good methods for approximating definite integrals. Taylor series are an effective tool for this kind of problem.

ex 1. Estimate $\int_0^1 \sqrt[5]{x^3+1}\,dx$.

Soln. I don't know how to find the antiderivative of $\sqrt[5]{x^3+1}$, but I can approximate its Taylor series @ $x=0$ using the binomial series.

$$f(x) = \sqrt[5]{x^3+1} = (1+x^3)^{\frac{1}{5}} \approx \underbrace{1 + \frac{1}{5}x^3 + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{2}(x^3)^2}_{\text{using binomial series w/ } p=\frac{1}{5} \text{ and substituting } x^3 \text{ for } x}$$

$$f(x) \approx 1 + \frac{1}{5}x^3 - \frac{2}{25}x^6$$

Let's use this degree 6 Taylor polynomial to estimate the integral.

$$\int_0^1 \sqrt[5]{x^3+1}\,dx \approx \int_0^1 1 + \frac{1}{5}x^3 - \frac{2}{25}x^6\,dx = x + \frac{1}{20}x^4 - \frac{2}{175}x^7\Big]_0^1$$

$$= 1 + \frac{1}{20} - \frac{2}{175} \approx 1.03857$$

For comparison, the actual value is

$\int_0^1 \sqrt[5]{x^3+1}\,dx \approx 1.04175$, so we're pretty close!

4. Computing Sums

Our last main application is for computing the values of infinite series.

ex. Compute $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{n!}$.

Soln. The trick is to notice that this looks like the Taylor series for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ with some number plugged in for x. It's slightly disguised, but no match for us.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} = e^{-4}$$ we did it!

ex 2. Compute $\sum_{n=0}^{\infty} \frac{(-1)^n e^7 \pi^{2n}}{(2n)!}$.

Soln. Those e's and π's look kind of scary but actually that e^7 is just there to throw you off. Notice that it doesn't have an n in the exponent. Let's factor that out.

$$\sum_{n=0}^{\infty} \frac{(-1)^n e^7 \pi^{2n}}{(2n)!} = e^7 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!}}$$ This reminds me of $\cos(x)$.

Recall that $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, so this series is $\cos(\pi) = -1$. Multiplying by e^7 we get our final answer:

$$\sum_{n=0}^{\infty} \frac{(-1)^n e^7 \pi^{2n}}{(2n)!} = -e^7.$$