SOLUTIONS BY THYDE.

6. [9 points]

a. [3 points] Find the first three nonzero terms in the Taylor series for $\dfrac{1}{\sqrt{1-x^2}}$ centered at $x = 0$.

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$$(1+x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n$$

$p = -\frac{1}{2}$, Binomial series

$$(1-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n x^{2n}$$

First 3 terms

$$= \binom{-\frac{1}{2}}{0} + \binom{-\frac{1}{2}}{1}(-1)^1 x^2 + \binom{-\frac{1}{2}}{2}(-1)^2 x^4$$

b. [4 points] Use your answer from part (a) to find the first three nonzero terms in the Taylor series for $\arcsin(2x)$ centered at $x = 0$. Recall that $\dfrac{d}{dx}\arcsin(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Take the antiderivative of the approximation from (a.)

$$\arcsin(x) = C + \binom{-\frac{1}{2}}{0}x + \frac{1}{3}\binom{-\frac{1}{2}}{1}(-1)x^3 + \frac{1}{5}\binom{-\frac{1}{2}}{2}(-1)^2 x^5$$

evaluate at $x=0$, $0 = \arcsin(0) = C$. Now evaluate at $2x$.

$$\boxed{\arcsin(2x) = 2\binom{-\frac{1}{2}}{0}x + \frac{2^3}{3}\binom{-\frac{1}{2}}{1}(-1)x^3 + \frac{2^5}{5}\binom{-\frac{1}{2}}{2}(-1)^2 x^5}$$

c. [2 points] Find the values of x for which the Taylor series from part (b) converges.

Binomial series converges for $|x| < 1$, so $\arcsin(x)$ does too.

Hence, $\arcsin(2x)$ converges for $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$.