team

tru X us

...dropping knowledg bombs from way atop that IVORY TOWER...

$\boxed{\text{tAkE sHeLtEr ImMeDiAteLy}}$

" Here we are, trapped in the amber of the moment. There is no why."

— the dearly deceased friend I never met; K.V.

ON A COMPUTER, YOU WILL FIND THAT WHICH YOU SEEK THIS WAY: ⇩

THOSE WITH FLORA, ⇨

Team Homework 8

① a) Let's first find the antiderivative of $(x^2+1)e^{-x}$.

- $\int (x^2+1)e^{-x}dx = (x^2+1)(-e^{-x}) + \underbrace{\int 2xe^{-x}dx}$

  $u = x^2+1 \qquad v = -e^{-x}$

  $du = 2x\,dx \qquad dv = e^{-x}dx$

  now we do this one.

- $\int 2xe^{-x}dx = -2xe^{-x} + 2\int e^{-x}dx = \underline{-2xe^{-x} - 2e^{-x}}$

  $u = 2x \qquad v = -e^{-x}$

  $du = 2dx \qquad dv = e^{-x}dx$

$\therefore \int (x^2+1)e^{-x}dx = -(x^2+1)e^{-x} - (2x)e^{-x} - 2e^{-x}$

$= -(x^2+2x+3)e^{-x}$.

Now, use definition of improper integral.

$$\int_0^\infty (x^2+1)e^{-x}dx = \lim_{b\to\infty}\int_0^b (x^2+1)e^{-x}dx = \lim_{b\to\infty} -(x^2+2x+3)e^{-x}\Big]_0^b$$

$$= \lim_{b\to\infty} 3 - \frac{(b^2+2b+3)}{e^b} = 3 - \lim_{b\to\infty}\frac{b^2+2b+3 \to\infty}{e^b \to\infty}$$ so we use L'hopital.

don't forget!

$$= \lim_{b\to\infty}\frac{2b+2 \to\infty}{e^b\to\infty}$$ and again

$$= \lim_{b\to\infty}\frac{2}{e^b} = 0,$$

So, $\boxed{\int_0^\infty (x^2+1)e^{-x}dx = 3.}$

b) we start w/ the antiderivative.

$$\int \frac{\ln(x)}{x^3}dx = -\frac{1}{2}\ln(x)x^{-2} + \frac{1}{2}\int \frac{1}{x^3}dx = \underline{-\frac{1}{2}\ln(x)x^{-2} - \frac{1}{4}x^{-2}}$$

$u = \ln(x)$ $\quad v = -\frac{1}{2}\frac{1}{x^2}$

$du = \frac{1}{x}dx$ $\quad dv = \frac{1}{x^3}dx$

Then use the definition. $\displaystyle\int_1^\infty \frac{\ln(x)}{x^3}dx = \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^3}dx$

$$= \lim_{b\to\infty} -\frac{1}{2}\ln(x)x^{-2} - \frac{1}{4}x^{-2}\Big]_1^b$$

$$= \lim_{b\to\infty} \frac{1}{4} - \frac{1}{2}\frac{\ln(b)}{b^2} - \frac{1}{4}\frac{1}{b^2}$$

make sure you see what I did here algebraically

$$= \frac{1}{4} - \lim_{b\to\infty}\frac{\frac{1}{2}\ln(b) + \frac{1}{4} \to \infty}{b^2 \to \infty} \quad \text{L'hopital}$$

don't forget her.

$$\lim_{b\to\infty}\frac{\frac{1}{2}\cdot 1/b}{2b} = \lim_{b\to\infty}\frac{1}{4b^2} = 0.$$

Therefore,

$$\boxed{\int_1^\infty \frac{\ln(x)}{x^3}dx = 1/4}$$

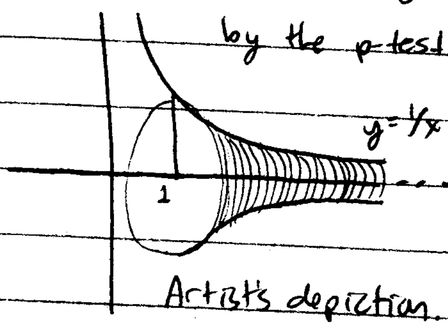
c) They tell us the antiderivative!

$$\int_1^\infty \frac{1-\frac{1}{2}\ln(x)}{x^{3/2}}dx = \lim_{b\to\infty}\int_1^b \frac{1-\frac{1}{2}\ln(x)}{x^{3/2}}dx = \lim_{b\to\infty}\frac{\ln(x)}{\sqrt{x}}\Big]_1^b = \lim_{b\to\infty}\frac{\ln(b) \to \infty}{\sqrt{b} \to \infty}$$

L'hopital $\Rightarrow$ $\displaystyle = \lim_{b\to\infty}\frac{1/b}{1/2\sqrt{b}} = \lim_{b\to\infty}\frac{2}{\sqrt{b}} = 0.$ wow, the areas perfectly cancel.

$$\boxed{\int_1^\infty \frac{1-\frac{1}{2}\ln(x)}{x^{3/2}}dx = 0}$$

② a) The area of the region we are revolving is infinite — this follo[ws] by the p-test. Seems pretty infinite to me, but mayb[e] life has a surprize for me....



Artist's depiction.

?

b) Throw-back: Solids of revolution → infinite style.

$$\int_1^{\infty} (\text{Area of cross-section})\, dx = \int_1^{\infty} \pi\left(\frac{1}{x}\right)^2 dx = \pi \int_1^{\infty} \frac{1}{x^2}\, dx.$$

radius.

Well, $\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2} dx = \lim_{b\to\infty} -1/x \Big]_1^b = \lim_{b\to\infty} 1 - \frac{1}{b} = 1.$

$$\boxed{\text{So, volume is } \pi \int_1^{\infty} \frac{1}{x^2} dx = \pi\ m^3}$$

c) WTF? This infinite obelisk has a finite volume even though the cross-sections have infinite area. May be I don't understand the nature of reality as well as I thought...

③ Here is the outline of our process:

1) Find the appropriate nice function $n(x)$

2) Check the limit condition.

3) Test the nice functions integral for convergence / divergence.

a) 1) $\overset{u(x)}{\overset{\shortparallel}{\dfrac{x^3+5x+1}{\sqrt{x^7+10000x^6+2}}}} \approx \dfrac{x^3}{\sqrt{x^7}} = \dfrac{1}{x^{7/2-3}} = \dfrac{1}{\sqrt{x}} = n(x)$

2) $\lim\limits_{x\to\infty} \dfrac{u(x)}{n(x)} = \lim\limits_{x\to\infty} \dfrac{x^{3.5}+5x^{1.5}+x^{.5}}{\sqrt{x^7+10000x^6+2}} = 1$ compare leading coeffs, same degree

$1 \neq 0, \infty$ ✓.

3) $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\,dx$ diverges b/c $p=\frac{1}{2}<1$ $\boxed{p\text{-test}}$.

$\therefore \displaystyle\int_1^\infty \frac{x^3+5x+1}{\sqrt{x^7+10000x^6+2}}\,dx$ diverges by $\boxed{\text{limit comparison test}}$

b) 1) $\overset{u(x)}{\overset{\shortparallel}{\dfrac{x-e^{-x}}{x^3+9}}} \approx \dfrac{x}{x^3} = \dfrac{1}{x^2} = n(x)$

2) $\lim\limits_{x\to\infty} \dfrac{u(x)}{n(x)} = \lim\limits_{x\to\infty} \dfrac{x^3-x^2e^{-x}}{x^3+9} = 1 \neq 0, \infty$ ✓.

3) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges b/c $p=2>1$ $\boxed{p\text{-test}}$

$\therefore \displaystyle\int_1^\infty \frac{x-e^{-x}}{x^3+9}\,dx$ converges by $\boxed{\text{limit comparison test}}$.

c) 1) My nice function is $n(x) = \frac{1}{x^2}$. How did I come up with this
Near 0, $\sin(x) \approx x$ (useful fact), so near ∞, $\sin(\frac{1}{x}) \approx \frac{1}{x}$.
This conclusion comes from a change of variables.
Hence, $\frac{\sin(\frac{1}{x})}{x} \approx \frac{1/x}{x} = \frac{1}{x^2}$.

How do we know $\sin(x) \approx x$ near 0? Check that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$
using L'hopital's rule.

useful trick

2) $\lim_{x \to \infty} \frac{u(x)}{n(x)} = \lim_{x \to \infty} x \sin(\frac{1}{x}) = \lim_{x \to \infty} \frac{\sin(1/x) \to 0}{(1/x) \to 0}$

L'hop it
$= \lim_{x \to \infty} \frac{\cos(1/x)\,(-1/x^2)}{(-1/x^2)} = \lim_{x \to \infty} \cos(\frac{1}{x}) \overset{\text{cos is continuous}}{=} \cos\left(\lim_{x \to \infty} \frac{1}{x}\right) = \cos(0) = 1.$

Wow! It works!

3) $\int_1^\infty \frac{1}{x^2}\,dx$ converges b/c $p = 2 > 1$ $\boxed{\text{p-test}}$

$\therefore \int_1^\infty \frac{\sin(1/x)}{x}\,dx$ converges by $\boxed{\text{limit comparison test}}$

Sorry my handwriting went to shit, this was a lot of writing.