TEAM Golden Eagle

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"The only thing I love more than separable differential equations, is my country" —tghyde

PRESENTS: The 7^{th} team homework, or
the best way to spend a spring afternoon on
your precious weekend.

THW #7

(i), (iii), and (iv) are separable, as we will demonstrate (ii) and (v) are not.

(i) $\frac{d\psi}{dy} = 5\psi - 4 \Longrightarrow \frac{d\psi}{5\psi - 4} = dy$

$$\Longrightarrow \frac{1}{5}\ln(5\psi - 4) = y + C_0$$

I changed the subscript on my constant each time I changed the constant.

$$\Longrightarrow \ln(5\psi - 4) = 5y + C_1$$

$$\Longrightarrow 5\psi - 4 = C_2 e^{5y}$$

$$\Longrightarrow \boxed{\psi = \frac{4}{5} + C_3 e^{5y} \quad \text{where } C_3 \text{ can be any number.}}$$

(iii) Let me abbreviate a few steps.

$$\frac{dp}{e^{2p}} = e^{3q}\,dq \Longrightarrow -\frac{1}{2}e^{-2p} = \frac{1}{3}e^{3q} + C_0$$

$$\Longrightarrow e^{-2p} = -\frac{2}{3}e^{3q} + C_1$$

$$\Longrightarrow -2p = \ln\left(C_1 - \frac{2}{3}e^{3q}\right)$$

$$p = -\frac{1}{2}\ln\left(C_1 - \frac{2}{3}e^{3q}\right)$$

we need $C_1 > 0$ for this function to be defined at all. Note $\ln(x)$ only defined for $x > 0$.

(iv) $\frac{dw}{5w} = \frac{dx}{(x+1)(x+2)} \Longrightarrow \frac{1}{5}\ln(w) + C_0 = \underbrace{\int \frac{dx}{(x+1)(x+2)}}_{\text{need partial fractions.}}$

Partial Fractions

$\frac{1}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} \Longrightarrow \frac{1}{x+2} = A + \frac{B(x+1)}{(x+2)}$

$\Longrightarrow$ evaluate @ $x=-1$

$\frac{1}{x+1} = \frac{A(x+2)}{(x+1)} + B$

$\boxed{1 = A.}$

$\Longrightarrow$ evaluate @ $x=-2$.

$\boxed{-1 = B.}$ So, $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$.

$\therefore \int \frac{dx}{(x+1)(x+2)} = \int \frac{1}{x+1} - \frac{1}{x+2}\, dx = \ln(x+1) - \ln(x+2)$.

log rules.

woops.

$\frac{1}{5}\ln(w) + C_0 = \cancel{\ln(x+1)(}\ \ln(x+1) - \ln(x+2) = \ln\left(\frac{x+1}{x+2}\right)$

$\Longrightarrow \boxed{w = C_1\left(\frac{x+1}{x+2}\right)^5}$ (follow that?) C_1 can be anything.

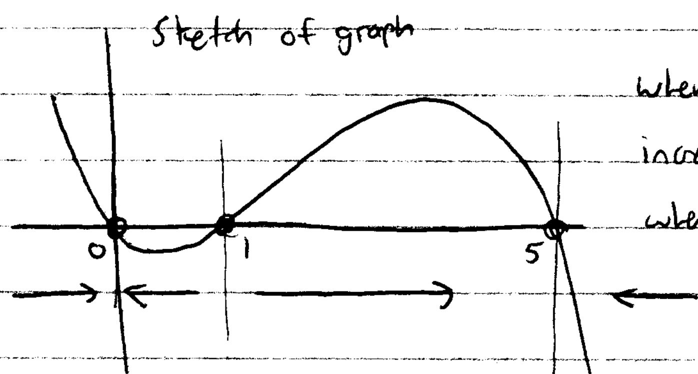
2 a) Constant population $\rightsquigarrow$ equilibrium solutions of the differential equation. $\rightsquigarrow$ values of P where $f(P) = 0$.

(Be sure you understand this!)

$\therefore$ $\boxed{P_0 = 0, 1, 5 \text{ from inspecting the graph.}}$

b) long-run behavior tends toward the nearest <u>stable equillibrium</u> or $\pm\infty$.

Let's first classify the equillibriums.



when the values of f are positive, P is increasing, and conversely are decreasing when f is negative. We summarize this information using the arrows.

Recall: stable equillibriums look like $\rightarrow\leftarrow$ and unstable equil. look like $\longleftrightarrow$. So, 0 and 5 are stable, 1 is unstable.

We summarize the long run behavior as follows:

- Any initial value in $(-\infty, 1)$ will tend toward 0.
- Any initial value in $(1, \infty)$ will tend toward 5.

(c) If he releases 99 toads, that corresponds to an initial value of .99. We saw above that this leads to total annihilation. If he buys 2 more toads, then he starts at $P = 1.01$ which leads to a population near 500 — a time of plenty.

Unstable equillibrims are pretty scary. 

③ $\dfrac{dM}{dt} = \begin{pmatrix}\text{rate of}\\ \text{money in}\end{pmatrix} - \begin{pmatrix}\text{rate of}\\ \text{money out}\end{pmatrix}$

Rate of money in: proportional to current net worth $\Longrightarrow kM$ for k the constant of proportionality.

Rate of money out: it is a constant rate 6.63 billion / year.

$\therefore \boxed{\dfrac{dM}{dt} = kM - 6.63}$

b) To solve for k, we need to use the information given about how his money grows: "In the absence of expenses, Calvin's wealth would double in 70 years".

If there were no expenses, we would have the diff'eq

$dM/dt = kM \implies M = M_0 e^{kt}$ (separation of variables)

Turn the sentence above into an equation: $M(70) = 2M_0$

$2M_0 = M_0 e^{k70} \implies 2 = e^{k70}$

$\implies \boxed{\frac{1}{70}\ln(2) = k}$

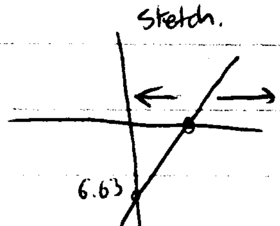
c) A stable equillibrium would be a net worth M_{stable} so that any nearby initial networth would tend toward it. Whereas an unstable equillibrium $M_{unstable}$ would "repel" any nearby networths. (just stating the definition of stability in this context)

d) Set original diff'eq equal to 0 and solve.

$kM - 6.63 = 0 \implies \boxed{M = \frac{6.63}{k} = \frac{70 \cdot 6.63}{\ln(2)}}$

Stable or unstable? Lets plot $kM - 6.63$ as a function of M.

It looks like a line. which direction does it face? Upward, because $k>0$



We make an "arrow diagram" and see that our equillibrium is $\boxed{\text{unstable}}$.

__Remark:__ this should make sense. If your income balances against your expenses, your money won't change. However, if your income increases above your expenses, you should

e) Solve the diff'eq by separation: $\dfrac{dM}{kM-6.63} = dt$

$$\Rightarrow \frac{1}{k}\ln(kM-6.63) = t + C_0$$

$$\Rightarrow kM - 6.63 = C_1 e^{kt}$$

$$\Rightarrow M = \frac{6.63}{k} + C_2 e^{kt}$$

Evaluate at $t=0$ to see how C_2 relates to M_0.

$$M_0 = \frac{6.63}{k} + C_2 \Rightarrow C_2 = M_0 - \frac{6.63}{k}$$

Now set $M_0 = 200$ to get

$$\boxed{M(t) = \frac{6.63}{k} + \left(200 - \frac{6.63}{k}\right)e^{kt} \quad \text{where } k = \frac{\ln(2)}{70}}$$