Math 116 — Practice for Exam 2

Generated February 24, 2016

Name: SOLUTIONS

Instructor: ______________________ Section Number: ________

1. This exam has 6 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

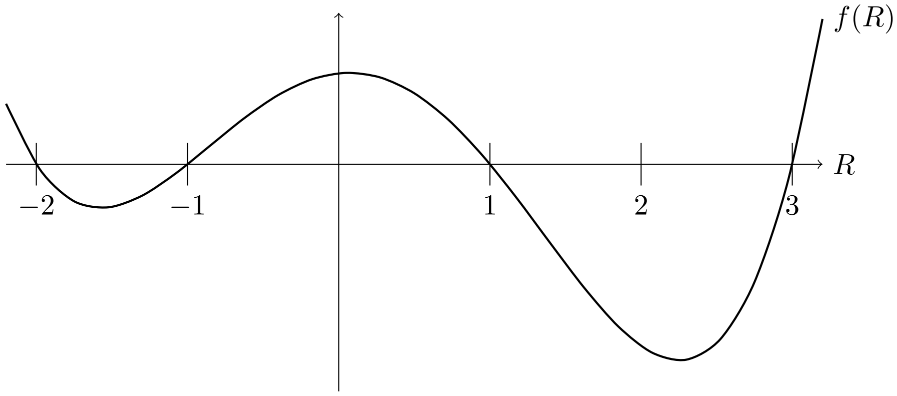
Semester	Exam	Problem	Name	Points	Score
Fall 2014	2	1	robot uprising	12	
Winter 2015	2	8		10	
Winter 2015	3	9	ladybugs1	10	
Winter 2013	3	6	drug	10	
Winter 2012	3	6	fish	9	
Fall 2012	2	2		14	
Total				65	

Recommended time (based on points): 67 minutes

1. [12 points] Franklin, your robot, is on the local news. Let $R(t)$ be the number of robots, in millions, that have joined the robot uprising t minutes after the start of the broadcast. After watching the news for a little bit, you find that $R(t)$ obeys the differential equation:

$$\frac{dR}{dt} = f(R)$$

for some function $f(R)$. A graph of $f(R)$ is shown below.



a. [3 points] If $R(t)$ is the solution to the above differential equation with $R(0) = 0$, what is $\lim_{t\to\infty} R(t)$? Justify your answer.

Solution: If $R = 0$, $f(R) = R'(t)$ is positive, so R will increase as t increases. As R increases to 1, $R'(t) = f(R)$ goes to 0, so $\lim_{t\to\infty} R(t) = 1$.

b. [6 points] Find the equilibrium solutions to the above differential equation **and** classify them as stable or unstable.

Solution:

$R = -2$	$\boxed{\textbf{Stable}}$	**Unstable**
$R = -1$	**Stable**	$\boxed{\textbf{Unstable}}$
$R = 1$	$\boxed{\textbf{Stable}}$	**Unstable**
$R = 3$	**Stable**	$\boxed{\textbf{Unstable}}$

c. [3 points] Let $R(t)$ be a solution to the given differential equation, with $R(3) = 0.5$. Is the graph of $R(t)$ concave up, concave down, or neither at the point $(3, 0.5)$? Justify your answer.

Solution:

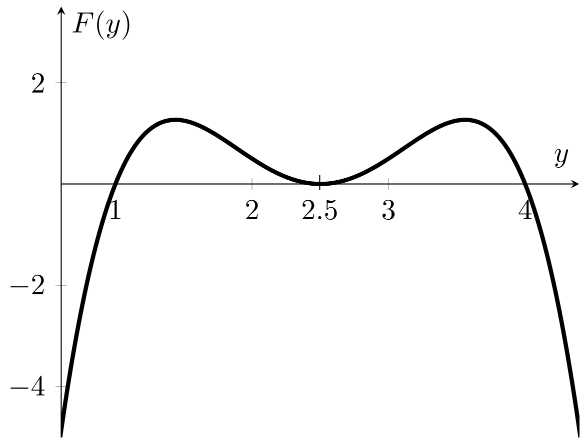
$$\frac{d^2R}{dt^2} = \frac{d}{dt}f(R) = f'(R)\frac{dR}{dt} = f'(R)f(R)$$

At $R = 0.5$, $f'(0.5) < 0$ and $f(0.5) > 0$ so $\frac{d^2R}{dt^2} < 0$. Therefore, the solution curve will be concave down.

8. [10 points] Consider the differential equation

$$\frac{dy}{dt} = F(y)$$

where $F(y)$ is graphed below.



a. [4 points] Identify all equilibrium solutions to the equation above.

Solution: We can find equilibrium solutions by setting $\frac{dy}{dt}$ to zero in the differential equation above and solving for y. In this case, this tells us that equilibrium solutions will be zeros of the function $F(y)$. From the graph, we then see that the equilibrium solutions will be $y = 1$, $y = 2.5$, and $y = 4$.

b. [4 points] Determine the stability of each equilibrium solution of the differential equation.

Solution: The equilibrium solutions $y = 1$ and $y = 2.5$ are unstable. The equilibrium solution $y = 4$ is stable.

c. [2 points] Suppose $y(t)$ solves the differential equation above subject to the initial condition $y(0) = 3$. Compute $\lim_{t\to\infty} y(t)$. Write your answer in the blank provided.

Solution:

$\lim_{t\to\infty} y(t) =$ ____4____

9. [10 points] Vic is planning to put ladybugs in his garden to eat harmful pests. The ladybug expert at the gardening store claims that the number of ladybugs in his garden can be modeled by the differential equation

$$\frac{dL}{dt} = \frac{L}{20} - \frac{L^2}{100}$$

where L is the number of ladybugs, in hundreds, in Vic's garden, t days after they are introduced.

a. [4 points] Find the equilibrium solutions to this differential equation and indicate their stability.

Solution: To find the equilibrium solutions we set the right hand side of the equation above equal to 0. The resulting equation simplifies to $L(5 - L) = 0$. There is then an unstable equilibrium at $L = 0$ and a stable equilibrium at $L = 5$.

b. [2 points] If Vic starts his garden with 50 ladybugs, what will the long term population of ladybugs in his garden be according to the differential equation above?

Solution: The long term population is 500 ladybugs, as the solution to the differential equation above with initial condition $L(0) = .5$ will tend to the stable equilbrium at $L = 5$ as $t \to \infty$.

The long term population is **500 ladybugs**

c. [4 points] For what value of b is the function $L(t) = 5e^{bt}\left(4 + e^{bt}\right)^{-1}$ a solution to this differential equation.

Solution: We can compute that $\frac{dL}{dt} = 5be^{bt}\left(4 + e^{bt}\right)^{-1} - 5e^{2bt}\left(4 + e^{bt}\right)^{-2} = \frac{20be^{bt}}{(1 + e^{bt})^2}$ and $\frac{L}{20} - \frac{L^2}{100} = \frac{5e^{bt}\left(4 + e^{bt}\right)^{-1}}{20} - \frac{25e^{2bt}\left(4 + e^{bt}\right)^{-2}}{100} = \frac{e^{bt}}{(1 + e^{bt})^2}$. These two expressions will be equal provided that $b = \frac{1}{20}$.

$b =$ **1/20**

6. [10 points] At a hospital, a patient is given a drug intravenously at a constant rate of r mg/day as part of a new treatment. The patient's body depletes the drug at a rate proportional to the amount of drug present in his body at that time. Let $M(t)$ be the amount of drug (in mg) in the patient's body t days after the treatment started. The function $M(t)$ satisfies the differential equation

$$\frac{dM}{dt} = r - \frac{1}{4}M \quad \text{with} \quad M(0) = 0.$$

a. [7 points] Find a formula for $M(t)$. Your answer should depend on r.

Solution: We use separation of variables

$$\frac{dM}{r - \frac{1}{4}M} = dt.$$

Using u-substition with $u = r - 1/4M, du = -1/4dM$ for the left-hand-side, we anti-differentiate:

$$-4 \ln |r - \frac{1}{4}M| = t + C_1.$$

Therefore,

$$\ln |r - \frac{1}{4}M| = -t/4 + C_2$$

and

$$|r - \frac{1}{4}M| = e^{-t/4+C_2} = C_3 e^{-t/4}.$$

Therefore

$$1/4M = r - C_3 e^{-t/4}$$

and

$$M(t) = 4r - C_4 e^{-t/4}.$$

With $M(0) = 0$, we conclude that $C_4 = 4r$, so we get $M(t) = 4r - 4re^{-t/4}$.

b. [1 point] Find all the equilibrium solutions of the differential equation.

Solution: $M = 4r$.

c. [2 points] The treatment's goal is to stabilize in the long run the amount of drug in the patient at a level of 200 mg. At what rate r should the drug be administered?

Solution: You need $4r = 200$, then $r = 50$ mg/day.

6. [9 points] Let $y(t)$ be the number of fish (**in hundreds**) in an artificial lagoon, where t is measured in years. The function $y(t)$ satisfies the following differential equation

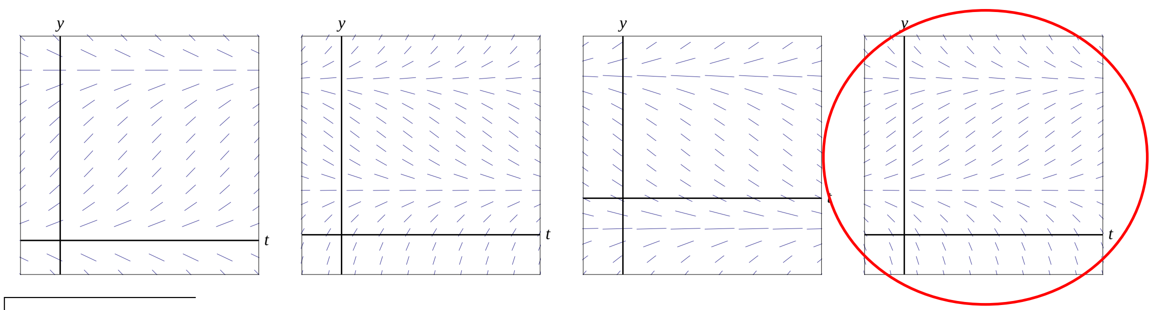
$$\frac{dy}{dt} = y\,(10 - y) - h.$$

where the constant h is the rate at which the fish are harvested from the lagoon.

a. [4 points] Suppose there is no harvesting ($h = 0$). Find the equilibrium solutions of the differential equation. Determine the stability of each equilibrium.

Solution: Equilibrium solutions are found by setting $\frac{dy}{dt} = 0$. So when $h = 0$, we have $y = 0$ and $y = 10$ as equilibrium solutions. Now when $y < 0$ or $y > 10$, we have $\frac{dy}{dt} < 0$. For $0 < y < 10$, we have $\frac{dy}{dt} > 0$. So $y = 0$ is an unstable equilibrium solution and $y = 10$ is a stable equilibrium solutions.

b. [2 points] Suppose the fish are harvested at a rate $h = 9$. Which of the following slope fields may correspond to the differential equation for $y(t)$? Circle your answer.



Solution: The equation is $y' = y(10 - y) - 9$. The equilibrium solutions are $y = 1$ (unstable) and $y = 9$ (stable).

c. [3 points] If (at $t = 0$) there are 200 fish in the lagoon, what is the maximum rate h for harvesting the fish, while still maintaining the fish population in the long run (i.e do not let the fish die out)? (Hint: You do not need to solve the differential equation to answer this question).

Solution: In order for the fish to not die out, we need $\frac{dy}{dt} \geq 0$. That gives us the equation $16 - h \geq 0$, so $h \leq 16$. Therefore 16 is the maximum rate of harvesting for the fish.

2. [14 points]

a. [10 points] Consider the following differential equations:

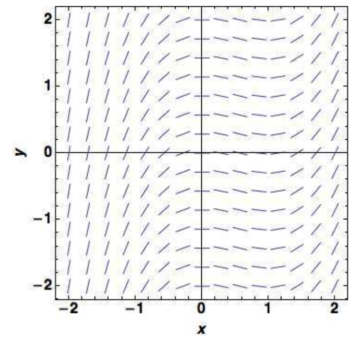
A. $y' = x(y-2)$ B. $y' = x(x-1)$ C. $y' = (x-y)y$ D. $y' = (2-y)(y+1)^2$

Each of the following slope fields belongs to one of the differential equations listed above. Indicate which differential equation on the given line. Find the equation of the equilibrium solutions and their stability. If a slope field has no equilibrium solutions, write none.

Differential equation: **B**

Equilibrium solutions and stability:

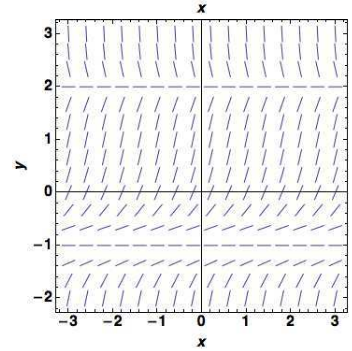
None



Differential equation: **D**

Equilibrium solutions and stability:

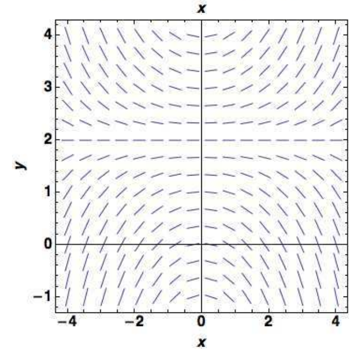
- $y = -1$ unstable (or semistable).
- $y = 2$ stable.



Differential equation: **A**

Equilibrium solutions and stability:

$y = 2$ unstable.



b. [4 points] Find the regions in the x-y plane where the solution curves to the differential equation $y' = (y - x^2)y$ are increasing.

Solution: $y' > 0$ if:

- $y > x^2$, or
- $y < 0$.