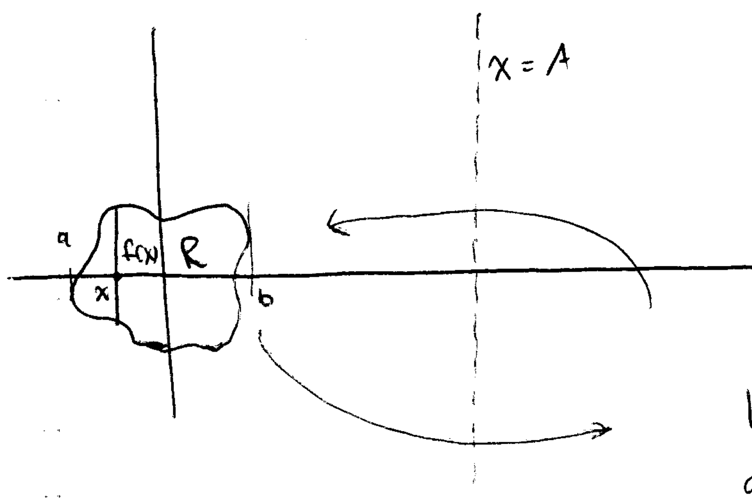
PRISCILLA'S FORMULA!

Consider some region R whose vertical cross-section at x has height $f(x)$. Let's revolve R around the line $x = A$. What is the volume of this object?



Priscilla says the volume should simply be

$$\boxed{\text{Area}(R) \cdot 2\pi|\bar{x} - A|}$$

where $\bar{x}$ is the x-coordinate center of mass of R.

In other words, we take the area of R and multiply it by the circumference of a circle whose radius is the distance between the axis and the center of mass of R.

Why is this true?
Let's derive it! Using shell method we see that the volume should be

$$\text{Vol.} = \int_a^b 2\pi|x - A|\,f(x)\,dx$$

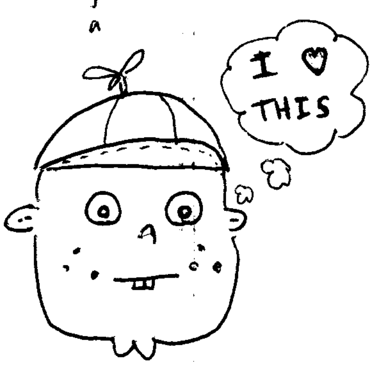
Let's suppose the region is to the left of $x = A$, so $|x - A| = A - x$.

Then we have

$$\text{vol} = \int_a^b 2\pi(A - x)\,f(x)\,dx = 2\pi A\int_a^b f(x)\,dx - 2\pi\int_a^b x\,f(x)\,dx \quad \text{(distribute!)}$$

$$= 2\pi \underbrace{\int_a^b f(x)\,dx}_{\text{Area}(R)} \left(A - \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}\right)$$

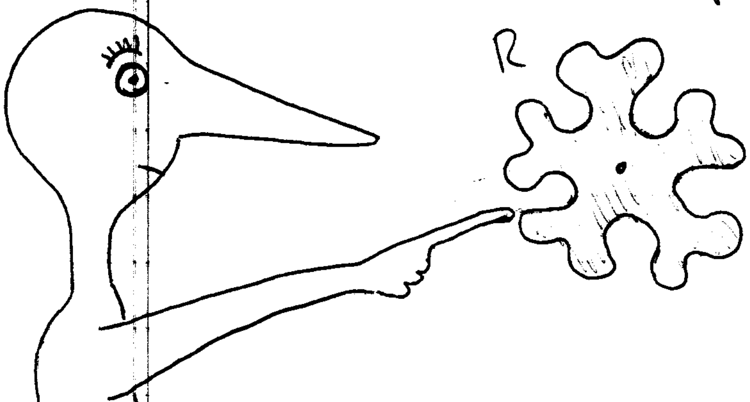
center of mass!



$$= 2\pi\,\text{Area}(R)\,(A - \bar{x})\,!$$

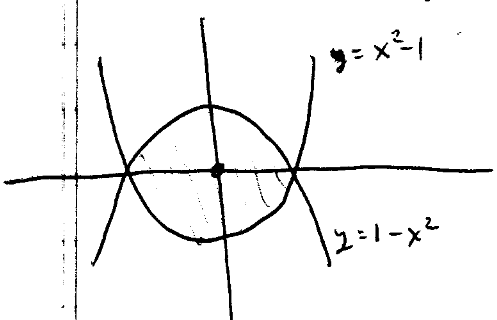
Awesome.

Example: Consider the crazy region R below and let's pretend it is as symmetric as I had tried to make it.



Then the center of mass is right in the middle of R. Hence, if we revolve R around ANY axis whose distance is L units from the center of mass, Then Priscilla's formula tells us that the volume will be $\text{Area}(R)\cdot 2\pi L$, so all we would need to do would be to compute the area of R, which we could maybe do with polar coordinates if we had a formula. Notice how much easier this is than trying to slice up R in some artificial way parallel to the axis of rotation.

Example: On Team Homework 2 you were asked to compute the volume of two rings formed by rotating the following region R around two different axes, $y=7$ and $x=7$



From the symmetry of R, we can see that the center of mass is at the origin, which is the same distance from both the axes. Hence, without any computation, we can conclude from Priscilla's formula that the volumes are the same! $\text{Vol} = \text{Area}(R)\cdot 2\pi\cdot 7$.

Let's compute Area(R)

$$\text{Area}(R) = 2\int_{-1}^{1} 1-x^2\,dx \underset{\uparrow \text{ even function}}{=} 4\int_0^1 1-x^2\,dx = 4\left(x-\tfrac{1}{3}x^3\right)\Big]_0^1 = \frac{8}{3}.$$

So, $\text{vol} = 2\pi\cdot 7\cdot \frac{8}{3}$!