Math 116 — Practice for Exam 2

Generated October 12, 2017

Name: SOLUTIONS

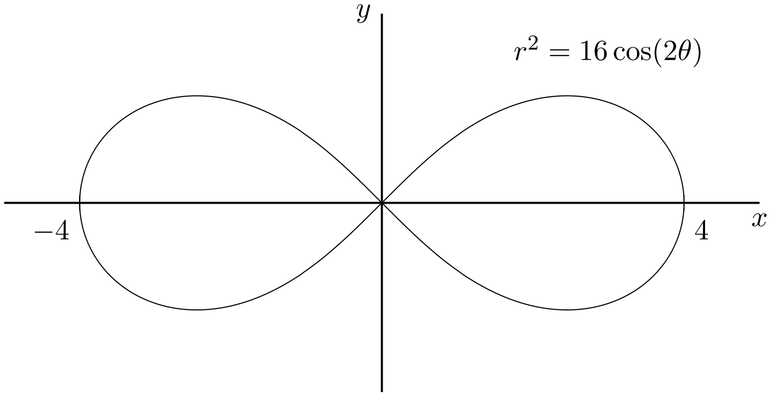
Instructor: ______________________________ Section Number: ____________

1. This exam has 3 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2017	2	2	lemniscate	12	
Fall 2016	2	7		9	
Fall 2016	2	1	propeller	9	
Total				30	

Recommended time (based on points): 27 minutes

2. [12 points] Chancelor was doodling in his coloring book one Sunday afternoon when he drew an infinity symbol, or lemniscate. The picture he drew is the polar curve $r^2 = 16\cos(2\theta)$, which is shown on the axes below. (The axes are measured in inches.)



a. [4 points] Chancelor decides to color the inside of the lemniscate red. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total area, in square inches, that he has to fill in with red.

Solution: Notice that we are given a formula for r^2 instead of r. Using symmetry, we will calculate the area in one quarter of the lemniscate and multiply by 4. To do this we will integrate from $\theta = 0$ to $\theta = \alpha$ where α is the smallest positive number for which $16\cos(2\alpha) = 0$. This gives $\alpha = \pi/4$. Using the formula for area inside a polar curve we see that the area is equal to $4 \cdot \frac{1}{2}\int_0^{\pi/4} 16\cos(2\theta)\,d\theta$ square inches.

b. [4 points] He decides he wants to outline the right half (the portion to the right of the y-axis) of the lemniscate in blue. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total length, in inches, of the outline he must draw in blue.

Solution: The portion of the lemniscate on the right of the y-axis corresponds to $-\pi/4 < \theta < \pi/4$. Notice that $\cos(2\theta) > 0$ for these angles.

Implicitly differentiating $r^2 = 16\cos(2\theta)$ (or directly differentiating $r = 4(\cos(2\theta))^{1/2}$), we find that $\dfrac{dr}{d\theta} = \dfrac{-4\sin(2\theta)}{\sqrt{\cos(2\theta)}}$ so $\left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{16\sin^2(2\theta)}{\cos(2\theta)}$.
Then using the arc length formula for polar coordinates we see that the length of the blue outline will be $\displaystyle\int_{-\pi/4}^{\pi/4}\sqrt{16\cos(2\theta) + \frac{16\sin^2(2\theta)}{\cos(2\theta)}}\,d\theta$ inches.

c. [4 points] Chancelor draws another picture of the same lemniscate, but this time also draws a picture of the circle $r = 2\sqrt{2}$. He would like to color the area that is inside the lemniscate but outside the circle purple. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total area, in square inches, that he must fill in with purple.

Solution: The circle and lemniscate intersect on the right side of the y-axis when $16\cos(2\theta) = 8$ or $\cos(2\theta) = \frac{1}{2}$. This gives angles $\theta_1 = -\pi/6$ and $\theta_2 = \pi/6$.

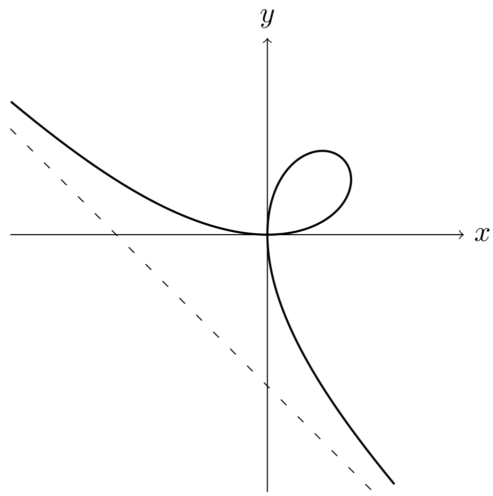
We first find the area between the curves on the right side using the formula for the area between polar curves and then multiply the by 2.

The resulting total area is $2 \cdot \frac{1}{2}\int_{-\pi/6}^{\pi/6}(16\cos(2\theta) - 8)\,d\theta$ square inches.

7. [9 points] For $-\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, consider the polar curve

$$r = \frac{\sin(2\theta)}{\cos(\theta) + \sin(\theta)}.$$

The curve has an asymptote, the dashed line in the picture, as θ approaches $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$.



a. [4 points] Write down, but do **not** evaluate, an integral that gives the area inside the loop in the first quadrant.

Solution: The area is given by

$$\frac{1}{2}\int_0^{\pi/2} \left(\frac{\sin(2\theta)}{\cos(\theta) + \sin(\theta)}\right)^2 d\theta.$$

b. [2 points] Find a formula for the quantity $x + y$ in terms of the variable θ. Write your answer in the space provided.

Solution:

$$x + y = \frac{\sin(2\theta)}{\cos(\theta) + \sin(\theta)}(\cos(\theta) + \sin(\theta)) = \sin(2\theta)$$

c. [2 points] Find the limit of $x + y$ as $\theta \to \left(\frac{3\pi}{4}\right)^-$. No justification is needed.

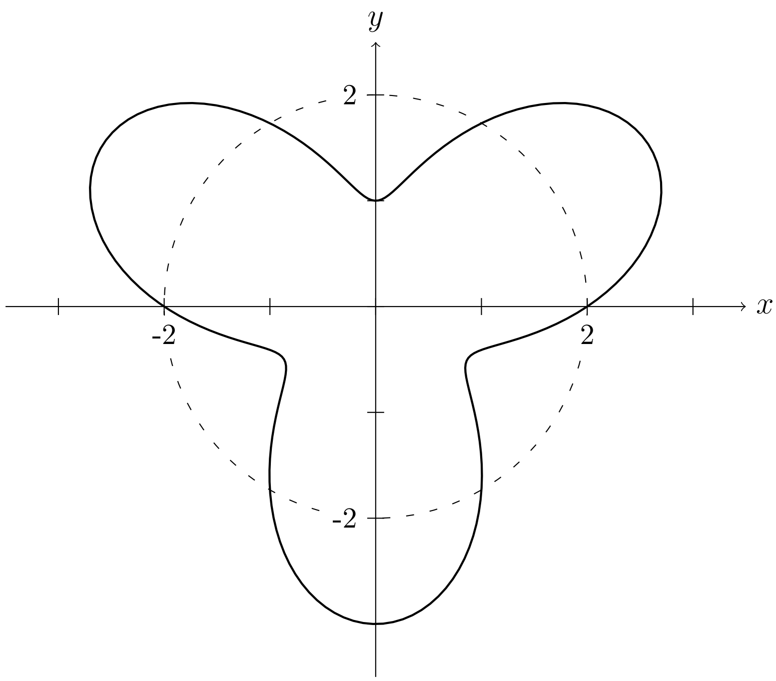
Solution: The specified limit is

$$\lim_{\theta \to (3\pi/4)^-} \sin(2\theta) = \sin\left(\frac{3\pi}{2}\right) = -1.$$

d. [1 point] Write down the Cartesian equation of the asymptote. No justification is needed.

Solution: The asymptote is given by $x + y = -1$.

1. [9 points] While assembling a large fan as a holiday gift, you discover the following lovely diagram in the instruction manual. The propeller and base of the fan are illustrated by the polar curve $r = 2 + \sin(3\theta)$ and the circle $r = 2$, respectively.



a. [5 points] Write down, but do **not** evaluate, an integral that gives the arc length of the part of the propeller **outside** the circle in the **first quadrant**.

Solution: The arclength is given by

$$\int_0^{\pi/3} \sqrt{(2+\sin(3\theta))^2 + (3\cos(3\theta))^2}\, d\theta,$$

or alternatively

$$\int_0^{\pi/3} \sqrt{(3\cos(3\theta)\cos(\theta) - (2+\sin(3\theta))\sin(\theta))^2 + (3\cos(3\theta)\sin(\theta) + (2+\sin(3\theta))\cos(\theta))^2}\, d\theta.$$

b. [4 points] Write down the Cartesian equation of the tangent line to the **propeller** at the point $(x, y) = (2, 0)$.

Solution: The tangent line is given by

$$y = \frac{2}{3}(x-2).$$