Math 116 — Practice for Exam 2

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Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 5 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2015	2	7	currency	18	
Winter 2015	2	6	caffeine drip	9	
Winter 2014	2	8		4	
Fall 2013	2	8	zombies	15	
Winter 2012	2	4	tank/bank	13	
Total				59	

Recommended time (based on points): 53 minutes

7. [18 points] A certain small country called Merrimead has 25 million dollars in paper currency in circulation, and each day 50 thousand dollars comes into Merrimead's banks. The government decides to introduce new currency by having the banks replace the old bills with new ones whenever old currency comes into the banks. Assume that the new bills are equally distributed throughout all paper currency. Let $M = M(t)$ denote the amount of new currency, in thousands of dollars, in circulation at time t days after starting to replace the paper currency.

a. [5 points] Write a differential equation involving $M(t)$, including an appropriate initial condition.

Solution: The concentration of old bills among all bills in circulation is $\frac{25000-M}{25000}$, and 50 thousand dollars moves through the bank each day, so

$$\frac{dM}{dt} = \text{Concentration} \times \text{Money per day} = \frac{25000-M}{25000} \cdot 50, \qquad M(0) = 0.$$

Now consider the differential equation

$$B^2 + 2B\frac{dB}{dt} = 2500.$$

b. [4 points] Find all equilibrium solutions and classify their stability.

Solution: If $\frac{dB}{dt} = 0$ then we see that $B^2 = 2500$, so the equilibrium solutions are $B = \pm 50$. Both equilibrium solutions are stable.

Brightcrest, a second small country, also wants to replace all of their old paper bills as well, using a different strategy than Merrimead. The amount $B(t)$, in millions of dollars, of new paper currency in circulation in Brightcrest at a time t years after starting to replace the paper currency is modeled by the differential equation for B above with initial condition $B(0) = 0$.

c. [6 points] Find a formula for $B(t)$.

Solution: Using separation of variables, we have $\int \frac{2B}{2500-B^2}\, dB = \int dt$. Using the substitution $w = 2500 - B^2, dw = -2B\, dB$, we see that $-\ln|2500 - B^2| = t + C$. Solving for B, we see $B = \sqrt{2500 - Ae^{-t}}$.
Since $B(0) = 0$, we see that $A = 2500$, so

$$B(t) = \sqrt{2500 - 2500e^{-t}}.$$

d. [3 points] Assuming that all of the old bills are replaced in the long run, how much time will pass after starting to replace the paper bills until the new currency accounts for 99% of all currency in Brightcrest?

Solution: Since all of the bills are replaced in the long run, the total amount of money in circulation is $\lim_{t\to\infty} B(t) = 50$ million dollars. So the amount of time that passes until the new bills account for 99% of all currency is the value of t so that

$$(.99)(50) = \sqrt{2500 - 2500e^{-t}}.$$

So $t = -\ln(1 - (.99)^2) \approx 3.92$ years.

6. [9 points] An extremely sleepy graduate student is grading Math 116 exams. She has been drinking coffee all day, but it just is not enough. She hooks up a caffeine drip that delivers caffeine into her body at a constant rate of 170 mg/hr. The amount of caffeine in her body decays at a rate proportional to the current amount of caffeine in her body. The half-life of caffeine in her body is 6 hours.

a. [4 points] Using the blank provided, write a differential equation which models the scenario described above. Use $Q(t)$ for the amount of caffeine in the graduate student's body, measured in mg, t for hours after she hooked up the caffeine drip, and $k > 0$ for the constant of proportionality.

Solution: The rate that the amount of caffeine in the graduate student's body is changing over time should be the rate that caffeine is entering their body minus the rate that caffeine is leaving their body. The rate that caffeine is entering the graduate students body is a constant 170 mg/hr. The rate that caffeine is leaving the graders body is proportional to the current amount, so it is kQ mg/hr. Putting all this together gives us the equation written below.

$$\frac{dQ}{dt} = \underline{\qquad 170 - kQ \qquad}$$

b. [5 points] Use the half-life of caffeine to determine the constant of proportionality.

Solution: We know that the amount of caffeine in the graduate student's body decays exponentially with decay rate k. If C_0 is the initial amount of caffeine, then a half-life of 6 hours means that

$$\frac{1}{2}C_0 = C_0 e^{-k6}.$$

Solving for k gives us that $k = -\frac{1}{6}\ln\left(\frac{1}{2}\right)$.

7. [7 points] Bill has just built a brand new 90,000 L swimming pool. Bill is allergic to chlorine so instead he is using a filtration system to prevent algae from building up in the pool. Algae grows in the pool at a constant rate of 600 kg/day. The filtration system receives a constant supply of 70,000 L/day of water and returns the water to the pool with 6/7ths of the algae removed. Let $A(t)$ be the amount of algae in the pool in kilograms t days after Bill has filled the pool with fresh (algae free) water.

a. [5 points] Write down the differential equation satisfied by $A(t)$. Include the initial condition.

Solution:
$\frac{dA}{dt}$ = Rate in-Rate out. Rate in $=$ 600 kg/day. Rate out=flow rate$\times$ concentration$\times$ fraction removed$= 70,000 \times \frac{A}{90{,}000} \times 6/7 = \frac{2}{3}A$. Thus $\frac{dA}{dt} = 600 - \frac{2}{3}A$.

$$\frac{dA}{dt} = 600 - \frac{2}{3}A$$

Initial condition: $A(0) = 0$

b. [2 points] Find all the equilibrium solutions of the differential equation.

Solution:
We want to solve $\frac{dA}{dt} = 600 - \frac{2}{3}A = 0$. Therefore $A = 900$ is the only equilibrium solution.

8. [4 points] Consider the differential equation $y' = e^y$. Solve the differential equation with initial condition $y(0) = 1$.

Solution:
The equation $\frac{dy}{dx} = e^y$ is separable so we have $e^{-y}dy = dx$. Integrating both sides we get $-e^{-y} = x + c$. Solving the equation we get $y = -\ln(c - x)$. To solve for c we take $y(0) = -\ln(c) = 1$. Therefore $c = \frac{1}{e}$. So the solution is $y = -\ln(\frac{1}{e} - x)$.

8. [15 points] Two zombies are chasing Jake down the Diag. Let $J(t)$ be Jake's position, measured in meters along the Diag, as he runs from the zombies. In this problem the time t is measured in seconds.

a. [3 points] The velocity of the first zombie is proportional to the difference between its own position, $S(t)$, and Jake's position, with constant of proportionality k. Using this fact, write the differential equation satisfied by $S(t)$.

Solution:

$$\frac{dS}{dt} = k(S - J(t)).$$

b. [2 points] State whether your equation in part (a) is separable. Circle the correct answer.

Solution:

The equation is: separable **NOT SEPARABLE**

Note: $J(t)$ is not constant, since Jake is running.

c. [9 points] The position of the second zombie at time t is given by the function $Z(t)$ (in meters), and satisfies the differential equation

$$\frac{dZ}{dt} = \alpha \frac{J(t)}{Z},$$

where α is a positive constant. Assuming that $Z(0) = 5$ and that Jake's position is given by $J(t) = 2t + 10$, find a formula for $Z(t)$.

Solution: Separating gives:

$$\int Z dZ = \alpha \int 2t + 10\ dt,$$

and so

$$\tfrac{1}{2}Z^2 = \alpha(t^2 + 10t) + C.$$

Plugging in $Z(0) = 5$, we see that $C = \frac{25}{2}$, so $Z(t)$ is given by:

$$Z(t) = \sqrt{2\alpha t^2 + 20\alpha t + 25}.$$

d. [1 point] In the differential equation $\dfrac{dZ}{dt} = \alpha \dfrac{J(t)}{Z}$, what are the units of α?

Solution: The units are m/s.

4. [13 points]

a. [6 points] A cylindrical tank with height 8 m and radius of 8 m is standing on one of its circular ends. The tank is initially empty. Water is added at a rate of 2 m^3 /min. A valve at the bottom of the tank releases water at a rate proportional to the water's depth (proportionality constant $= k$). Let $V(t)$ be the volume of the water in the tank at time t, and $h(t)$ be the depth of the water at time t.

i. Find a formula for $V(t)$ in terms of $h(t)$. $V(t) =$ ______________________________

ii. Find the differential equation satisfied by $V(t)$. Include initial conditions.

Solution: i) The formula is $V(t) = 64\pi h(t)$.

ii) The differential equation is

$$\frac{dV}{dt} = 2 - kh.$$

So now we can solve $h(t) = \frac{V(t)}{64\pi}$. Substituting in V for h, we get

$$\frac{dV}{dt} = 2 - k\frac{V}{64\pi}$$

with initial condition $V(0) = 0$.

b. [7 points] Let $M(t)$ be the balance in dollars in a bank account t years after the initial deposit. The function $M(t)$ satisfies the differential equation

$$\frac{dM}{dt} = \frac{1}{100}M - a.$$

where a is a positive constant. Find a formula for $M(t)$ if the initial deposit is $1,000$ dollars. Your answer may depend on a.

Solution: This equation is separable:

$$\frac{dM}{M - 100a} = \frac{1}{100}dt.$$

Integrating, we find $\ln|M - 100a| = \frac{t}{100} + C$. So we get

$$M = Be^{t/100} + 100a.$$

Using the initial conditions, $M(0) = 1000$, so $1000 = B + 100a$. Substituting back in we get

$$M = 100\left((10 - a)e^{t/100} + a\right).$$