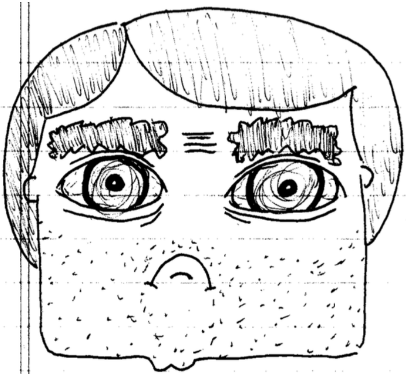
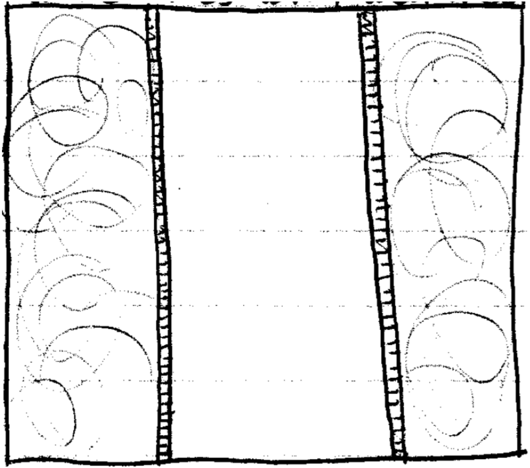


How does a parabola look on the horizon?

Trevor Hyde
October 6th, 2016

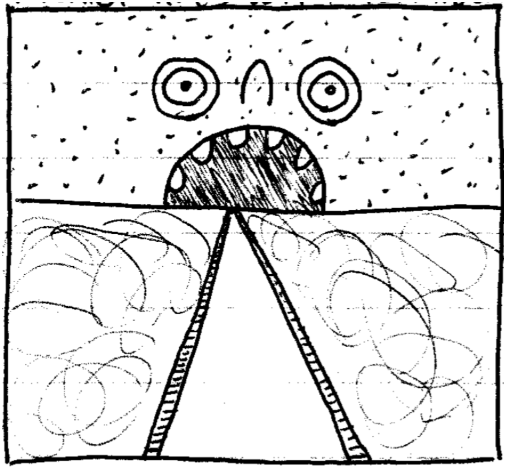
Parallel Lines

Coplanar lines that never intersect.



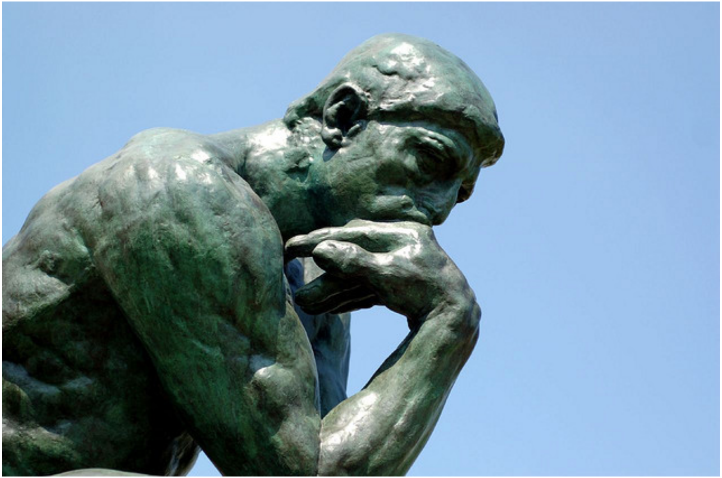
Parallel Lines

Appear to intersect on the horizon.



$\Longrightarrow$Weird stuff can happen on the horizon.

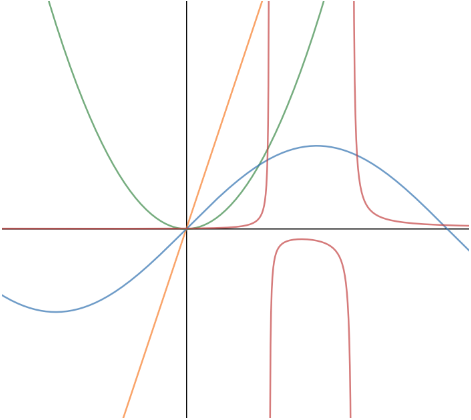
Makes you wonder...



What else is going on out there on the horizon?

Fixed Perspective

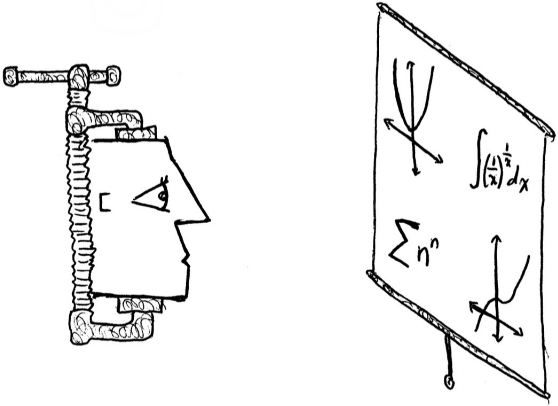
In our studies, we see lots of curves and graphs of functions...



...but always from the same top-down point of view.

Plato’s Cave

It’s almost like we’ve been studying in Plato’s cave our whole life.

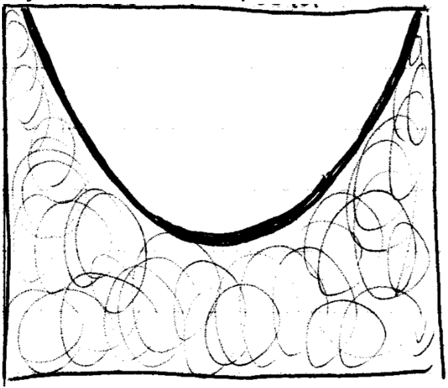


So my question for you is...

ARE YOU READY TO BREAK THE CHAINS???

Consider the Parabola

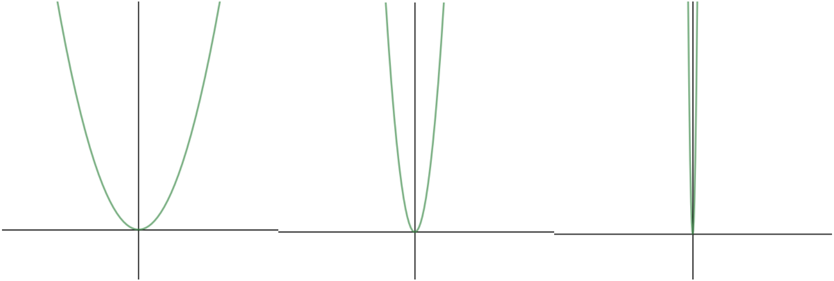
Everybody “knows” that a parabola looks like this:



But does it *really*?

Consider the Parabola

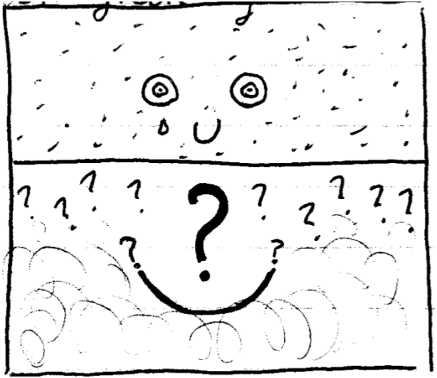
If we zoom out, it quickly loses the familiar bowl shape.



Try to imagine what your life would be like if you'd only ever seen parabolas from far away.

Consider the Parabola

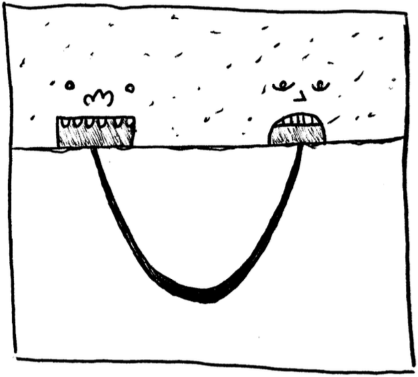
But we're still just looking straight down.
What would it look like if we cast our gaze upwards and looked out toward the horizon?



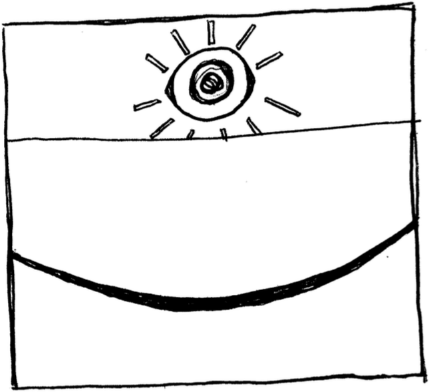
Will it intersect the horizon?
How many times?

Consider the Parabola

Does it look like this?



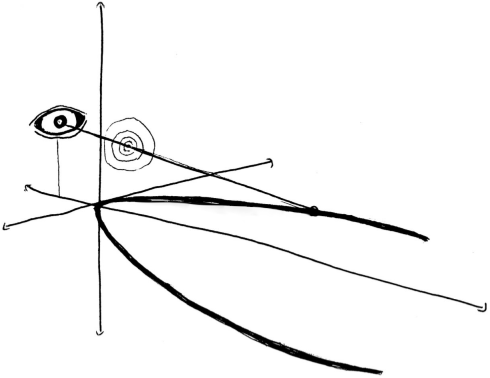
Or maybe like this?



Let’s find out.

Computing What We See

Graph the parabola $y = x^2$ on the $z = 0$ plane.
Say your eye is at $\langle 0, -1, 1 \rangle$.
Draw a line from your eye to each point on the graph,
$\Longrightarrow$ mark where it intersects the $y = 0$ plane.
This will give a picture of what we see.



Computing What We See

A point on the parabola looks like $\langle x, x^2, 0\rangle$.

$L(t) = (1-t)\langle 0, -1, 1\rangle + t\langle x, x^2, 0\rangle = \langle tx, t(1+x^2) - 1, 1-t\rangle$.

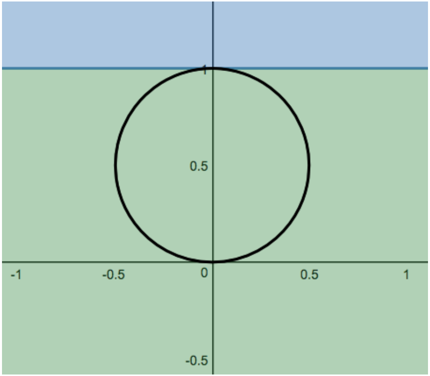
Find intersection with $y = 0$:

$$t(1+x^2) - 1 = 0 \Longrightarrow t = \frac{1}{1+x^2}$$

Plug back into $L(t)$ to get the point $\left\langle \frac{x}{1+x^2}, 0, \frac{x^2}{1+x^2} \right\rangle$.

$y = x^2$ on the Horizon

$$\left\langle \frac{x}{1+x^2}, 0, \frac{x^2}{1+x^2} \right\rangle$$

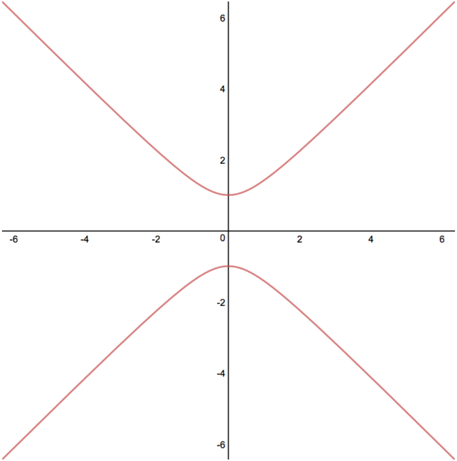


It looks like a circle tangent to the horizon!

Q: What significance does the center of this circle have in terms of the graph of the parabola?

What About a Hyperbola?

$$y^2 - x^2 = 1$$

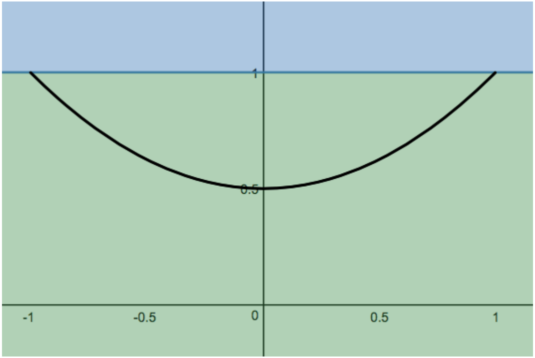


What will this look like on the horizon?

Hyperbola on the Horizon

Using the same approach we parametrize the image of the top half of the hyperbola,

$$\left\langle \frac{x}{1+\sqrt{1+x^2}}, 0, \frac{\sqrt{1+x^2}}{1+\sqrt{1+x^2}} \right\rangle.$$



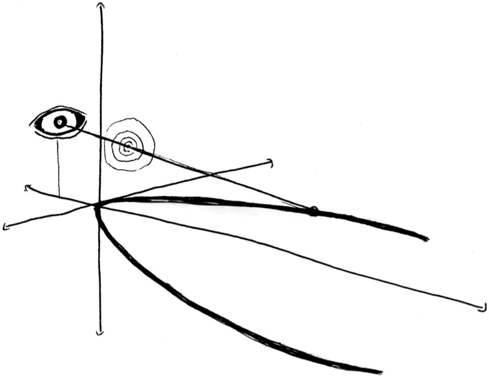
Old Putnam Problem

1. Can you cover the plane with the interiors of finitely many parabolas?

2. Can you cover the plane with the interiors of finitely many hyperbolas?

Other Curves?

What do the graphs of other functions $y = f(x)$ look like on the horizon?



Using the same approach we can solve the general problem.

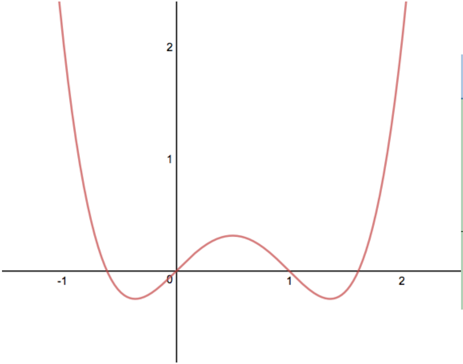
Behold!

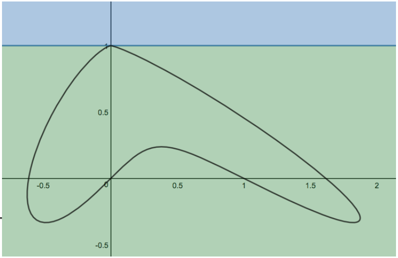
$$\left\langle \frac{x}{1+f(x)}, 0, \frac{f(x)}{1+f(x)} \right\rangle$$

If $y = f(x)$ is the graph of a polynomial, how it looks on the horizon depends on whether the degree is **even** or **odd**.

Even Degree Polynomial

Consider $y = x^4 - 2x^3 + x$.

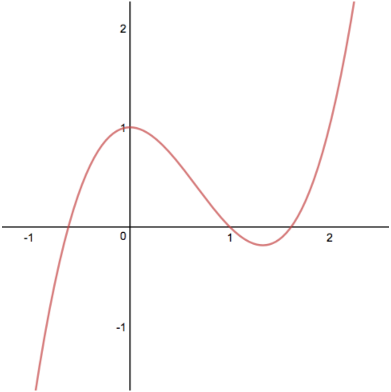


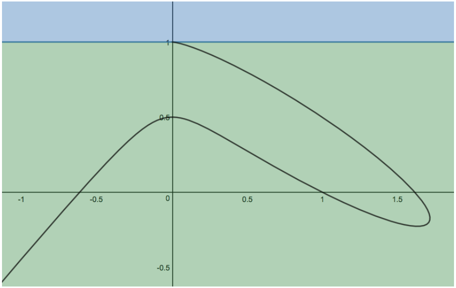


Always appear tangent to horizon at the same point.*

Odd Degree Polynomial

Consider $y = x^3 - 2x^2 + 1$.

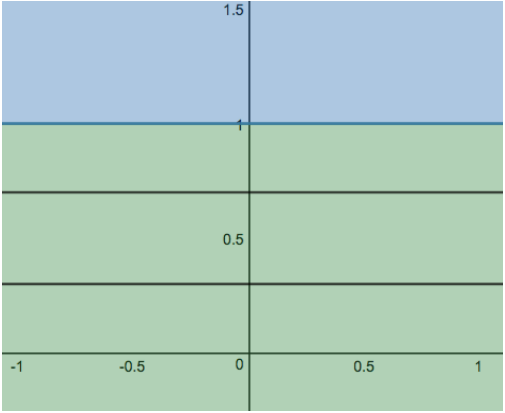




Always appear tangent to horizon at the same point.*

*Caveat

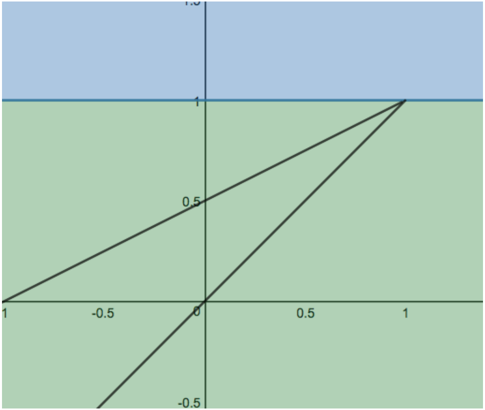
Degree 0 polynomials are constant functions,



they are parallel to the horizon.

*Caveat

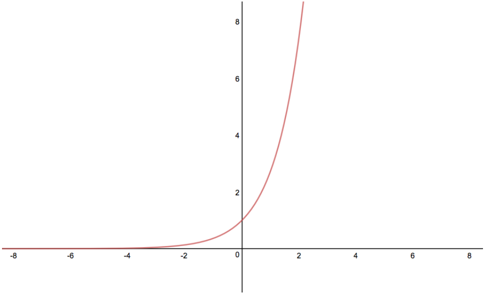
Degree 1 polynomials are linear functions,

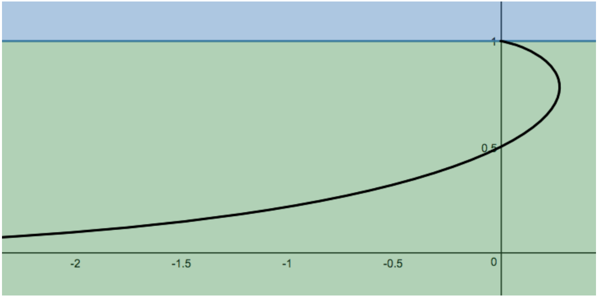


where they intersect the horizon depends on their slope.

Exponential Functions

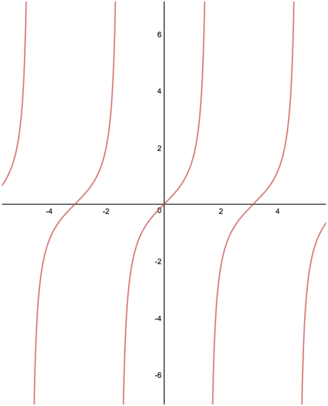
Two views of $y = e^x$.

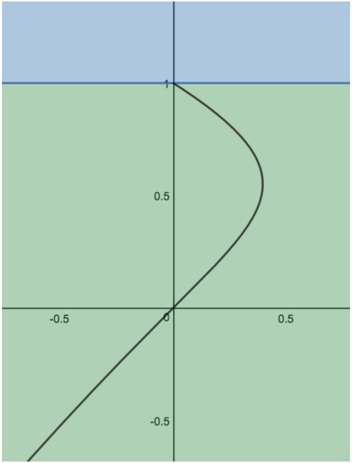




Tangent

$$y = tan(x).$$

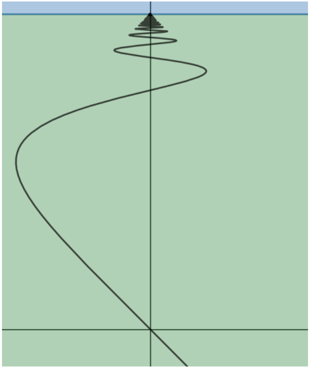


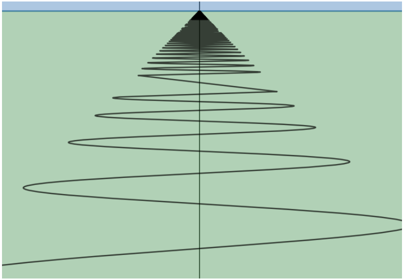


Looks like one period of the sine function!

Sine

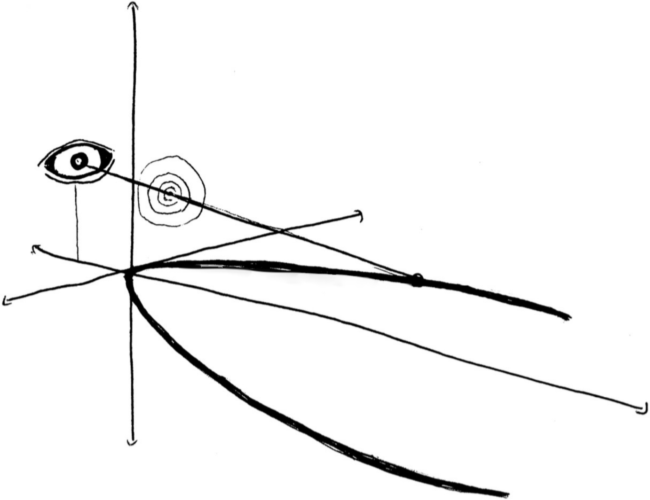
The sine function $y = sin(x)$ is bounded, so it doesn't go to the horizon in the positive y direction.
But we can still turn and see what it looks like in the positive x direction.





Working Backwards

If we start with an image we would like to see from a given vantage point, we can work backwards to see how it would appear on the ground.



Working Backwards

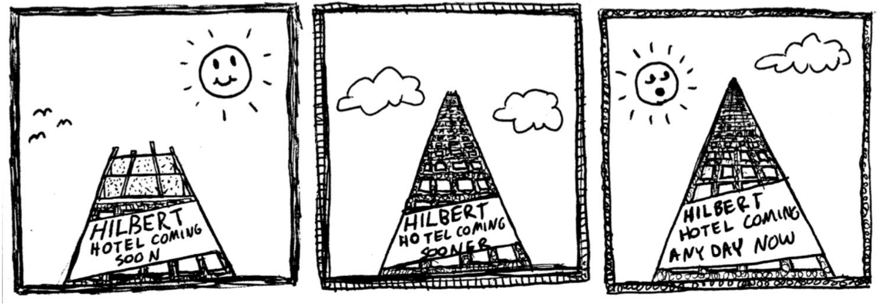
This is how cool 3D street art is made.

Infinity in 3D

Is there a horizon in three dimensional space?

Touch the Sky

Suppose hotel construction begins along your walk to school.
Each day the tower gets taller.



The tower seems to converge to a particular **point** in the sky.
That **point** is not a place you can go,
it lives on the **horizon sphere**!

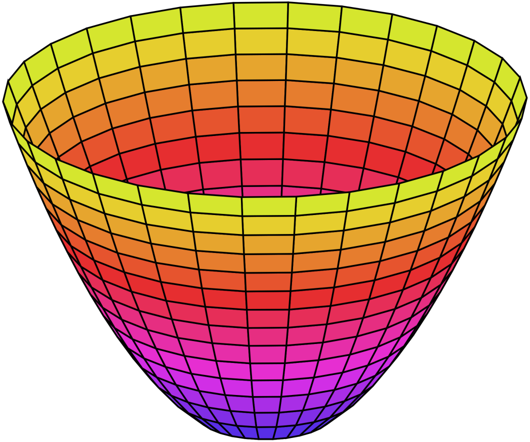
Horizon Sphere

Parallel lines appear to converge to the same point in the sky.
We can imagine these points on a sphere *infinitely far away*.
Point to a spot on the sphere with an **astronomy laser pointer**.



How does a paraboloid look on the horizon sphere?

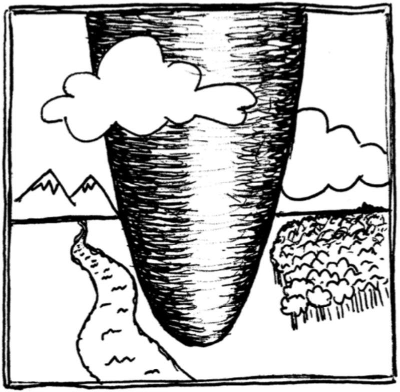
A **paraboloid** is a surface whose cross-sections in two directions are parabolas pointing in the same direction.



For example, the graph of $z = x^2 + y^2$.

Wild Paraboloid

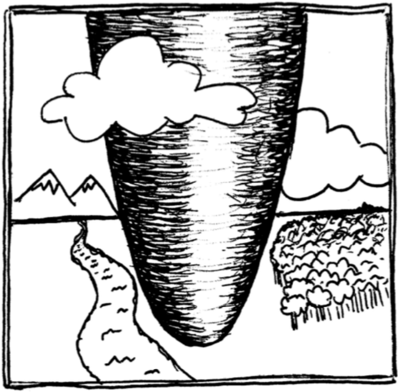
Let's try to imagine what it would look like to encounter a paraboloid in the wild.



It's pretty tall.

Wild Paraboloid

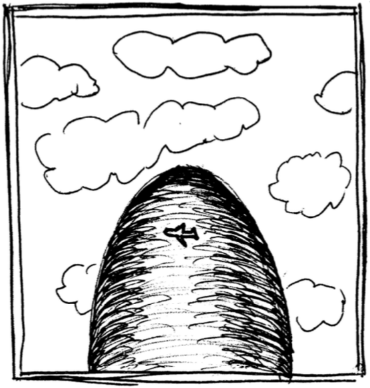
What will it look like near the horizon sphere?



No matter where you're standing on the plane, there is a point on the paraboloid directly above you.

Paraboloid on the Horizon

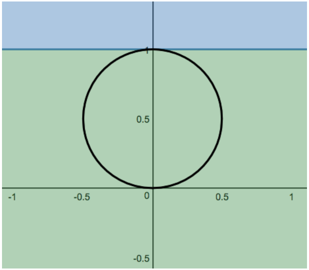
Here's my best sketch of what you would see if you looked straight up.



Let's try to understand why.

Building a Paraboloid

One way to construct the paraboloid $z = x^2 + y^2$ is by revolving the parabola $z = x^2$ around the z-axis.
(You probably want to build it lying down and then stand it up.)
We already worked out how parabolas look near the horizon.



Just revolve it.

Questions

1. How do hyperboloids look near the horizon sphere?
2. Can you cover space with the interiors of finitely many paraboloids? hyperboloids?
3. What does a saddle surface look like near the horizon sphere?
4. What does it all *mean*???

Thanks!

(Please clap.)