DAY 23

$\sum_{n=1}^{\infty} a_n$ converge? ($\sum a_n$ is shorthand for $\sum_{n=1}^{\infty} a_n$)

Test 1: n^{th} term test.

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

Test 2: Integral test.

- f(x) decreasing
- f(x) positive

$\int_1^{\infty} f(x)dx$ converges $\iff$ $\sum_{n=1}^{\infty} f(n)$ converges.

Test 3: p-test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

Test 4: Comparison

$0 \leq a_n \leq b_n$

$\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

$\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges.

Test 5: LCT

$a_n, b_n > 0$

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0, \infty$ (samesies)

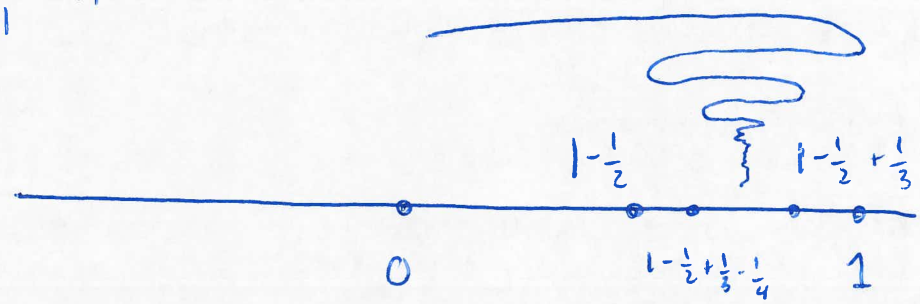
then $\sum a_n$ converges $\iff$ $\sum b_n$ converges.

Test 6: Geometric series test.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \underline{\text{if}} \quad \underbrace{|x| < 1}_{\star}$$

TIME FOR MORE TESTS!

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln(2) \text{ btw.}$$



So this sum must converge!

Test 7: Alternating Series Test.

If a_n is a sequence such that

1. (positive) $a_n \geq 0$
2. (decreasing) $a_n > a_{n+1}$
3. (goes to 0) $\lim_{n\to\infty} a_n = 0$

$\bigstar$

Then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges. ← can be $(-1)^n$ or $(-1)^{n+1}$

ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ $\quad a_n = \frac{1}{n}$

1. $\frac{1}{n} > 0$
2. $\frac{1}{n} > \frac{1}{n+1}$
3. $\lim_{n\to\infty} \frac{1}{n} = 0$

You need to show this work.

So $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by alt. series test.

ex. Tricky.

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^2} \longrightarrow$$

n	1	2	3	4
$\cos \pi n$	-1	1	-1	1

$$\parallel \qquad \cos(\pi n) = (-1)^n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \longleftarrow \text{alt series.}$$

$a_n = \frac{1}{n^2}$

1. $\frac{1}{n^2} > 0$
2. $\frac{1}{n^2} > \frac{1}{(n+1)^2}$
3. $\lim_{n\to\infty} \frac{1}{n^2} = 0.$

so $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2}$ converges by alt series test.

Test Test 8: Abs. conv. test

If $\sum |a_n|$ converges, then $\sum a_n$ conv.

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$$

yuck.

changes sign erradically.

can't use comparison — need positive terms.

$$\sum \left|\frac{\sin(n)}{n^2}\right| = \sum \frac{|\sin(n)|}{n^2} \le \sum \frac{1}{n^2}$$

converges by p test

converges by comparison.

$0 \le |\sin(n)| \le 1$

$0 \le \frac{|\sin(n)|}{n^2} \le \frac{1}{n^2}$

$\therefore \sum \frac{\sin(n)}{n^2}$ converges by abs conv. test

If $\sum |a_n|$ converges, say $\sum a_n$ converges <u>absolutely</u>

If $\sum a_n$ converges but $\sum |a_n|$ diverges, say $\sum a_n$ is <u>conditionally conv.</u>

For example, $\sum \frac{\sin(n)}{n^2}$ is abs. conv.

$\sum \frac{(-1)^{n+1}}{n}$ is conditionally conv.

Note: you can rearrange a conditionally conv. sum to get <u>ANY VALUE</u>.

TEST 9: RATIO TEST.

Let a_n be a sequence, suppose

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

If $L < 1$, then $\sum a_n$ converges

If $L > 1$, then $\sum a_n$ diverges

If $L = 1$, then ???

ex. $$\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$$

$a_n = \frac{1}{n!}$ $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{(n+1)!} \frac{n!}{1} = \lim_{n\to\infty} \frac{1}{n+1} = 0. < 1$$

So $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges! by ratio test.

Factorials + ~~ex~~ exponential $\rightsquigarrow$ ratio test.

ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{3^n(2n)!}$

$a_n = \dfrac{(n!)^2}{3^n(2n)!}$ $\quad \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!^2}{3^{n+1}(2(n+1))!} \frac{3^n(2n)!}{(n!)^2}$

$\displaystyle = \lim_{n\to\infty} \frac{3^n}{3^{n+1}}\left(\frac{(n+1)!}{n!}\right)^2 \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{1}{3}\frac{(n+1)^2}{1}\frac{1}{(2n+2)(2n+1)} = \frac{1}{12} < 1$

so converges by ratio test.