Reduction Stability and Iterate Decomposition Stability

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July 25, 2016
Let $K = \overline{K}$, $\text{char}(K) = 0$.

Let $f, u$ be non-constant rational functions defined over $K$ with $\deg f \geq 2$.

$$f^n(x) = u(y)$$

$C_n$ arise in the study of the dynamical Mordell-Lang problem.

Is $C_n$ irreducible? What can we say about the components of $C_n$?

For each $n$ we have a finite map

$$C_{n+1} \rightarrow C_n$$

$$(x, y) \mapsto (f(x), y)$$
\( u_n : C_n \rightarrow \mathbb{P}^1 \) defined by \( u_n(x, y) = x \).

Total degree of the projection \( u_n \) is \( \deg u \).

Restricting \( u_n \) to irreducible components gives a partition of \( \deg u \).

Hence the branching must eventually stabilize.
Question

- How long does it take for the $C_n$ to stabilize?
- Can we have a situation like this for large $n$?

\[ C_{n-3} \rightarrow C_{n-2} \rightarrow C_{n-1} \rightarrow C_n \rightarrow C_{n+1} \]
Theorem (H, Zieve) Let $K = \overline{K}$, $\text{char}(K) = 0$. Suppose $f, u$ are non-constant rational functions defined over $K$ such that $\deg f \geq 2$.

- (RS) There exists a constant $b = b(\deg u)$ such that if $C_b : f^b(x) = u(y)$ is irreducible, then $C_n$ is irreducible for all $n \geq 0$.

- (RS') There exists a constant $b' = b'(\deg u)$ such that for all $n \geq b'$, $C_n$ has the same number of irreducible components as $C_{b'}$.

- (IDS) There exists a constant $b'' = b''(\deg u)$ such that if $f^n = u \circ v$ for some $n \geq 1$ and rational function $v$, then $f^{b''} = u \circ w$ for some rational function $w$.

RS' follows from RS by induction.
\( f^n = u \circ v \) iff \( C_n : f^n(x) = u(y) \) has a genus 0 component of the form \( y = v(x) \) iff \( C_n \) has a component \( D \) for which the \( x \)-coordinate projection \( u_n : D \to \mathbb{P}^1 \) has degree 1.

RS’ provides \( b' \) so that \( C_{b'} : f^{b'}(x) = u(y) \) must have genus 0 component for which the \( x \)-coordinate projection has degree 1.
Theorem (Fried) Let $g, h$ be non-constant rational functions defined over a field $K$. If $g(x) = h(y)$ is reducible, then we have

$$g = g_0 \circ g_1$$

$$h = h_0 \circ h_1$$

such that $g_0, h_0$ have the same Galois closure and $g_0(x) = h_0(y)$ is reducible.

- Suppose $C_n : f^n(x) = u(y)$ were reducible. Let $u = u_0 \circ u_1$ and $f^n = f_0 \circ f_1$ be the decompositions given by Fried’s theorem.
- $u_0$ and $f_0$ having same Galois closure implies $\deg f_0 \leq \deg u_0! \leq \deg u!$.
- IDS provides $b''$ so that $f^{b''} = f_0 \circ f_2$ for some $f_2$.
- Then $f_0(x) = u_0(y)$ reducible implies $C_{b''} : f^{b''}(x) = f_0(f_2(x)) = u_0(u_1(y)) = u(y)$ reducible.
Using Fried’s theorem we reduce to the case where $C_b : f^b(x) = u(y)$ is irreducible of genus 0.

Riemann-Hurwitz argument to show that if $b \geq \log \left( (2 + 1/42) \deg u \right) / \log(2)$, then the $x$-projections $u_i : C_i \to \mathbb{P}^1$ have Galois closure of genus at most 1 for $i \leq b/2$ and $\# \{p : p$ is a critical value of $u_i$ for some $i \leq b/2\} \leq 4$.

Rational functions $u(y)$ with Galois closure of genus at most 1 can be classified up to change of coordinates.
• $u(y)$ is, after a change of coordinates, either $y^m, y^m + y^{-m}, \pm T_m(y)$, or one of finitely many functions with Galois group $A_4, S_4, \text{or } A_5$; or comes from an isogeny of elliptic curves (for example, Lattès maps.)

• In each case, knowing the ramification of $u$ and assuming $C_b$ is irreducible of genus 0, R-H limits the possible ramification of $f$ over the critical values of $u$.

• If $b$ is sufficiently large, the ramification of $f$ is constrained enough that we can classify all possibilities.

• But then we conclude in each case that $C_n$ is always irreducible.
Theorem (H, Zieve) Let $B, C$ be projective curves defined over an algebraically closed field $K$ of characteristic 0. Suppose

\[ u : C \to B \]
\[ f : B \to B \]

are finite morphisms defined over $K$ such that $\deg f \geq 2$.

- (RS) There exists a constant $b = b(\deg u)$ such that if the fiber product $C_b$ of $f^b$ and $u$ is irreducible, then $C_n$ is irreducible for all $n \geq 0$.

- (RS’) There exists a constant $b' = b'(\deg u)$ such that for all $n \geq b'$, the fiber product $C_n$ of $f^n$ and $u$ has the same number of irreducible components as $C_{b'}$.

- (IDS) There exists a constant $b'' = b''(\deg u)$ such that if $f^n = u \circ v$ for some $n \geq 1$ and $v : B \to C$, then $f^{b''} = u \circ w$ for some $w : B \to C$. 
Thank you!

These slides may be found on my website:

www-personal.umich.edu/~tghyde/