Orbits Visiting Finite Sets

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Joint work with Mike Zieve
Visiting a finite set

The set-up:

- $K$ be a field
- $f(x) \in K(x)$ be a rational function
- $p \in K$ be a point
- $S \subseteq K$ a finite set.

When does the $f$-orbit of $p$ visit the finite set $S$?

$$\{n : f^n(p) \in S\} = ?$$
When does the $f$-orbit of $p$ visit $S$?

- Typically finitely often, probably never.
- However, if the $f$-orbit of $p$ visits $S$ more than $|S|$ times, then it does so infinitely often!

$\{n : f^n(p) \in S\} = \text{finite union of arithmetic progressions}$
Visiting a finite set again

Let’s make this question more interesting by replacing all iterates of a single rational function

$$\langle f \rangle = \{f^n : n \geq 0\}$$

with all words in a finite set of rational functions

$$M = \langle f_1, f_2, \ldots, f_m \rangle = \{f_{i_1}f_{i_2} \cdots f_{i_k} : k \geq 0\}$$

$M$-orbit of $p = M(p) = \{w(p) : w \in M\}$.

When does the $M$-orbit of $p$ visit the finite set $S$?

$$\{w \in M : w(p) \in S\} = ?$$
When does the $M$-orbit of $p$ visit $S$?

**Theorem (H, Zieve)**

Let $K$ be a field and let $M = \langle f_1, \ldots, f_m \rangle$ with $f_k(x) \in K(x)$ such that $\deg(f_k) \geq 2$. If $p \in K$ and $S \subseteq K$ is a finite set, then

$$\{ w \in M : w(p) \in S \}$$

is a *regular language*. 
A **regular expression** is a type of pattern used to describe a collection of words ( = sequences of letters from an alphabet.)

\{Regular expressions\} is the closure of the alphabet under

- concatenation \((w₁w₂)\)
- disjunction \(w₁|w₂\)
- Kleene star \(w^*\)

**Ex.** Say our alphabet consists of two letters \(f\) and \(g\).

- \((f|g)^*f\) describes all words “starting” with \(f\)
- \((f^*gf^*gf^*)^*\) describes all words with an even number of \(g\)’s

A **regular language** is the collection of all words described by a regular expression.
Theorem (H, Zieve)

Let $K$ be a field and let $M = \langle f_1, \ldots, f_m \rangle$ with $f_k(x) \in K(x)$ such that $\deg(f_k) \geq 2$. If $p \in K$ and $S \subseteq K$ is a finite set, then
\[
\{ w \in M : w(p) \in S \} \text{ is a regular language.}
\]

Ex. Let $p = 2$, $S = \{1, 4\}$, and $M = \langle f, g \rangle$ where
\[
f(x) = x^2, \quad g(x) = \frac{-11x^3 + 57x^2 - 70x + 24}{24}.
\]
Then \{ $w \in M : w(p) \in S$ \} is the regular language described by
\[
(f^*gf^*gf^*)^*g \mid (fg)^*f
\]
Preperiodic points

Say \( p \in K \) is **preperiodic** under \( f(x) \) if for some \( j \geq 0, k \geq 1 \)

\[ f^{j+k}(p) = f^j(p). \]

► What does it mean for a point \( p \) to be preperiodic under a finitely generated dynamical system \( M = \langle f_1, f_2, \ldots, f_m \rangle \)?
$p$ preperiodic under $f \iff f$-orbit of $p$ is finite.

Say $p$ is preperiodic under $M = \langle f_1, f_2, \ldots, f_m \rangle$ if the $M$-orbit of $p$ is finite.
Ex: Let $M = \langle f, g \rangle$ where

$$f(x) = x^2 \quad g(x) = x^2 - 1.$$ 

Then 0 is preperiodic under $M$. 
Finite orbits of $\langle f \rangle$ all have the same “shape”.

But there is a wide range of finite orbit shapes for $M = \langle f_1, f_2, \ldots, f_m \rangle$.

These are deterministic finite automata (DFA)!
Deterministic finite automata (DFA)

DFA over an alphabet $f_1, f_2, \ldots, f_m$ is a finite directed graph $A$ with a distinguished start state $p$ and set $S$ of accept states.

- vertices = states
- labelled edges = transitions.
- For each letter $f_k$ there is exactly one transition labelled $f_k$ out of each state.
Kleene’s Theorem

An automata $A$ is a simple machine for processing words.
- Beginning at the start state $p$ read $w$ one letter at a time and transition accordingly.
- If we end at a state in $S$, then $A$ accepts $w$.
- Language of $A$ is the set of all words $L(A)$ accepted by $A$.

Theorem (Kleene’s Theorem)

- If $A$ is a DFA, then $L(A)$ is a regular language.
- If $L$ is a regular language, then there is a DFA $A$ such that $L = L(A)$. 
Theorem (Kleene’s Theorem)

- If $A$ is a DFA, then $L(A)$ is a regular language.
- If $L$ is a regular language, then there is a DFA $A$ such that $L = L(A)$.

**Ex.** The regular language $L$ described by the regular expression

$$(f^*gf^*g)^*(f^*gf^*) = \text{ all words with an odd number of } g\text{'s},$$

is accepted by $A$ shown below with $S = \{q\}$. 

![Diagram of a DFA]
Theorem (H, Zieve)

Let $K$ be a field and let $M = \langle f_1, \ldots, f_m \rangle$ with $f_k(x) \in K(x)$ such that $\deg(f_k) \geq 2$. If $p \in K$ and $S \subseteq K$ is a finite set, then

$$\{w \in M : w(p) \in S\}$$

is a regular language.
Proof sketch

- Let $h$ be a height function on $\mathbb{P}^1(K)$
- $(\deg(f_k) \geq 2)$ There exists a $B$ such that
  - $h(s) \leq B$ for $s \in S$ or $s = p$,
  - $h(f_k(q)) > h(q)$ whenever $h(q) > B$.
- Let $A$ be the finite automaton with states consisting of
  - all $q \in K$ with $h(q) \leq B$
  - a “dead state” $D$
- Transition labelled $f_k$ from $q$ to $D$ iff $h(f_k(q)) > B$.
  - $D$ only transitions to itself.
- $L(A) = \{w \in M : w(p) \in S\}$. 

Further Questions

- Given $M = \langle f_1, f_2, \ldots, f_m \rangle$ can we characterize the automata $A$ for which there are $A$-periodic points?
  - Using interpolation, all automata possible for some choice of rational functions.

- If $M$ has good reduction at a prime $\ell$ and $p$ is an $A$-periodic point, how does the period of $p$ modulo $\ell$ relate to $A$?
  - Does this lead to new dynamical unit constructions?

- Suppose that $f$ is a continuous endomorphism of a real interval $X$. Sharkovskii proved that if $f$ has a 3-periodic point in $X$, then it has periodic points of all periods in $X$.
  - Does this generalize to the non-cyclic setting?
Thank you!