Exploring the Topograph
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Binary Quadratic Forms (BQF)

- **Binary quadratic form:**
  Degree 2 homogeneous polynomial in 2 variables.
  
  \[ f(x, y) = ax^2 + bxy + cy^2 \]

- **Ex.** \( f(x, y) = 3x^2 - 2xy + y^2 \).

- Seems easy?
Popular with old-timey Europeans like Fermat, Lagrange, Legendre, Euler, Gauss.

Classic questions:

- Which primes are the sum of two squares?
  \[ p = x^2 + y^2? \]
- When is a BQF invariant under change of coordinates?
  \[ f(x, y) = x^2 - xy + y^2 = f(x - y, x). \]
- When are two BQF the same after change of coordinates?
What’s the Big Deal?!

Legendre writing about BQF in 1798.

David A. Cox writing about BQF in 1989.

A primitive positive definite form $ax^2 + bxy + cy^2$ is said to be reduced if

$$|b| \leq a \leq c, \text{ and } b \geq 0 \text{ if either } |b| = a \text{ or } a = c.$$  

(Note that $a$ and $c$ are positive since the form is positive definite.) The basic theorem is the following:

**Theorem 2.8.** Every primitive positive definite form is properly equivalent to a unique reduced form.

**Proof.** The first step is to show that a given form is properly equivalent to one satisfying $|b| \leq a \leq c$. Among all forms properly equivalent to the given one, pick $f(x, y) = ax^2 + bxy + cy^2$ so that $|b|$ is as small as possible. If $a < |b|$, then

$$g(x, y) = f(x + my, y) = ax^2 + (2am + b)xy + c'y^2$$

is properly equivalent to $f(x, y)$. Since $a < |b|$, we can choose $m \in \mathbb{Z}$ so that $|2am + b| < |b|$, which contradicts our choice of $f(x, y)$. Thus $a \geq |b|$, and $c \geq |b|$ follows similarly. If $a > c$, we need to interchange the outer coefficients, which is accomplished by the proper equivalence $(x, y) \mapsto (-y, x)$. The resulting form satisfies $|b| \leq a \leq c$.

The next step is to show that such a form is properly equivalent to a reduced one. By definition (2.7), the form is already reduced unless $b < 0$ and $a = -b$ or $a = c$. In these exceptional cases, $ax^2 - bxy + cy^2$ is reduced, so that we need only show that the two forms $ax^2 \pm bxy + cy^2$ are properly equivalent. This is done as follows:

- $a = -b : (x, y) \mapsto (x + y, y)$ takes $ax^2 - axy + cy^2$ to $ax^2 + axy + cy^2$.
- $a = c : (x, y) \mapsto (-y, x)$ takes $ax^2 + bxy + ay^2$ to $ax^2 - bxy + ay^2$.

The final step in the proof is to show that different reduced forms cannot be properly equivalent. This is the uniqueness part of the theorem. If $f(x, y) = ax^2 + bxy + cy^2$ satisfies $|b| \leq a \leq c$, then one easily shows that

$$|f(x, y)| \geq (a - |b| + c) \min(x^2, y^2)$$

(see Exercise 2.7). Thus $f(x, y) \geq a - |b| + c$ whenever $xy \neq 0$, and it follows that $a$ is the smallest nonzero value of $f(x, y)$. Furthermore, if $c > a$, then $c$ is the next smallest number represented properly by $f(x, y)$, so that in this case the outer coefficients of a reduced form give the minimum values properly represented by any equivalent form. These observations are due to Legendre [74, Vol. 1, pp. 77–78].
Where are the primitive vectors in the topograph?

If $\pm e_1$ is in a base $\{\pm e_1, \pm e_2, \pm e_3\}$ which is in turn in a superbase $\{\pm e_1, \pm e_2\}$, then $\pm e_1$ is in just one of the other two bases in the superbase, namely $\{\pm e_1, \pm e_2\}$. So in our picture, in which we have suppressed the $\pm$'s, the nodes and edges that involve $e_1$ form a path. We can therefore add a face bounded by this path to our topograph and identify it with $\pm e_1$ (so that the picture becomes more like a travel map on a surface). In the resulting fully labelled topograph, each region is labelled with a (lax) vector $\pm v$ (but we usually omit the sign), two vectors separated by an edge form a (lax) base, and three around a vertex form a (lax) superbase.

Norms of vectors

Up to this point in our discussion of the topograph, the values of $f$ haven’t even been mentioned. (So we see that the shape of the topograph does not depend on $f$.) We now fix on a particular quadratic form $f$ and, for this $f$, call $f(v)$ the norm of $v$.

The arithmetic progression rule

Suppose we know the values of a quadratic form $f$ at the three vectors $\{e_1, e_2, e_3\}$ of some superbase. How do we find its values elsewhere? We use the formula

$$f(v_1 + v_2) + f(v_1 - v_2) = 2[f(v_1) + f(v_2)],$$

which is essentially equivalent to a well-known geometrical theorem of Apollonius. To verify this, let $B(v_1, v_2)$ be the bilinear form asso-
Introducing: the Topograph
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Topographical Facts

- No vector ever repeats.
- Vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ are adjacent if and only if
  \[ ad - bc = \text{det} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \pm 1. \]
Say $\begin{bmatrix} a \\ b \end{bmatrix}$ is **primitive** if $\gcd(a, b) = 1$.

- Every primitive vector shows up in the topograph.
- Define the **Top** and **Bottom** operations by
  $$ T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a + b \\ b \end{bmatrix} \quad B \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a + b \end{bmatrix}. $$

- Every primitive vector can be made from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ using a sequence of top and bottom operations.
**Topographical Facts**

**Idea:** work backwards!

\[
\begin{bmatrix}
3 \\
11
\end{bmatrix} = B^3 \begin{bmatrix}
3 \\
2
\end{bmatrix} = B^3 T^1 \begin{bmatrix}
1 \\
2
\end{bmatrix} = B^3 T^1 B^2 \begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

Can you see why this works? Try more examples!
Idea: work backwards!

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Can you see why this works? Try more examples!
BQF on the Topograph

\[ f(x, y) = ax^2 + bxy + cy^2 \]

- \[ f(kx, ky) = a(kx)^2 + b(kx)(ky) + c(ky)^2 = k^2 f(x, y) \]
- \[ f(-x, -y) = f(x, y) \]
- Enough to know the values of \( f(x, y) \) on primitive vectors.
BQF on the Topograph
$f(x,y) = x^2 + y^2$
The Arithmetic Progression Law

\[ f(v_1 - v_2) \]

\[ f(v_1) + f(v_2) \]

\[ f(v_1 + v_2) \]
We don’t even need to know the vectors!

Coordinate free way to study BQF.
From (Almost) Nothing to Everything!
From (Almost) Nothing to Everything!
From (Almost) Nothing to Everything!
From (Almost) Nothing to Everything!
Formula Found

\[ f(x,y) = ax^2 + hxy + cy^2 \]
The Climbing Lemma
The Climbing Lemma

Keeps Climbing
Race to the Bottom

- We can always head downhill!

- What happens at the bottom?
Stuck in the Well

- **Climbing Lemma**: At most one well.
- If $f > 0$, then it has a well and $f$ is called **positive definite**.
  - To tell if two positive definite forms are equivalent, go meet at the well.
Climbing Lemma: At most one river!

If \( f \neq 0 \) and has a river, then \( f \) is called **indefinite.**
Climbing Lemma: At most one river!

If $f \neq 0$ and has a river, then $f$ is called **indefinite**.
Discrimination

If $a$ and $c$ are adjacent values of $f$ with common difference $h$ between them, then the **discriminant** of $f$ is

$$\Delta = h^2 - 4ac.$$ 

- Same value measured anywhere on the topograph!
- **Ex.** $\Delta = -8$. 

`\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{topograph.png}
\end{figure}`
River or Well?

Well: \( \Delta < 0 \)

\[
\Delta = 0^2 - 4 \cdot 1 \cdot 2 = -8
\]

River: \( \Delta > 0 \)

\[
\Delta = 1^2 - 4 \cdot 3 \cdot (-2) = 25
\]
Suppose $f$ has a river.

- Finitely many options for $a$, $c$, $h$.
- Therefore the river must always repeat!
The Repeating River

\[ f(x, y) = x^2 + 3xy - 2y^2 \]
If $f$ has a lake, then $\Delta = h^2$ is a square.

One lake $\implies f(x, y) = ax^2$. 
If both ± appear, there are two lakes connected by a river.
Sometimes the river floods and the lakes merge.
A **symmetry** of a BQF $f$ is a linear change of coordinates that leaves $f$ invariant,

$$f(Ax + By, Cx + Dy) = f(x, y).$$

Alternatively, it is a transformation of the topograph that preserves the $f$ labelling.
Symmetries

- A **symmetry** of a BQF $f$ is a linear change of coordinates that leaves $f$ invariant,

  $$f(Ax + By, Cx + Dy) = f(x, y).$$

- Alternatively, it is a transformation of the topograph that preserves the $f$ labelling.
Symmetries

Symmetries preserve all special features!

- Wells, rivers, and lakes, are preserved.
Another Perspective

\[ \gamma_0 = \infty \]

[Diagram of a circle with labels]
Another Perspective

The mediant of reduced fractions $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a + c}{b + d}$.
Another Perspective

The **mediant** of reduced fractions $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$.
Equivalence
Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent if and only if their difference is an Egyptian fraction $\frac{1}{m}$.

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} = \pm \frac{1}{bd}.$$
Every reduced fraction eventually appears.

- Define Top and Bottom operations on fractions by

\[ T\left(\frac{a}{b}\right) = \frac{a + b}{b} \quad \quad B\left(\frac{a}{b}\right) = \frac{a}{a + b}. \]

- Notice: T is “add one” and B is “flip, add one, flip.”

\[ T\left(\frac{a}{b}\right) = 1 + \frac{a}{b} \quad \quad B\left(\frac{a}{b}\right) = \frac{1}{\frac{a+b}{a}} = \frac{1}{1 + \frac{b}{a}} = 1 + \frac{1}{\frac{a}{b}}. \]

- Applying T and B repeatedly is easy.

\[ T^m\left(\frac{a}{b}\right) = m + \frac{a}{b} \quad \quad B^m\left(\frac{a}{b}\right) = \frac{1}{m + \frac{1}{\frac{a}{b}}}. \]

- (flip, add one, flip)(flip, add one, flip) = (flip, add two, flip)
Recall that
\[
\begin{bmatrix}
3 \\
11
\end{bmatrix} = B^3 T^1 B^2 \begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

Check it out:
\[
B^3 T^1 B^2 \left( \frac{1}{0} \right) = \frac{1}{3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{\infty}}}} = \frac{1}{3 + \frac{1}{1 + \frac{1}{2}}} = \frac{1}{3 + \frac{2}{3}} = \frac{3}{11}
\]
Continued Fractions

Continued fraction expansion of \( \frac{a}{b} \) is a path from \( \infty \) to \( \frac{a}{b} \):

\[
\frac{3}{11} = 3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\infty}}}
\]
But wait, there’s more!

New model shows there are hidden cells in our topograph living off near the horizon.

\[ \sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ldots}}}} \]
Back to the River

\[
f(x, y) = x^2 - 2y^2 \quad \Rightarrow \quad \frac{f}{y^2} = \left(\frac{x}{y}\right)^2 - 2
\]

**Lagrange:** If \( d \geq 0 \) and \( a, b, d \) are integers, then the continued fraction expansion of \( a + b\sqrt{d} \) is eventually periodic.
Topographical Trinitarianism
The End

Thanks!
Want More?

*The Sensual (Quadratic) Form* by John H. Conway.

*Topology of Numbers* by Allen Hatcher.