PROOF OF FTC.

First, the theorem:

<u>Fundamental Theorem of Calculus:</u>

Let $F(t)$ be differentiable on $[a,b]$, then
(with $F'(t)$ continuous)

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

Proof: Let's start with the definition of the integral.

$$\int_a^b F'(t)\,dt = \lim_{n\to\infty} LEFT(n) = \lim_{n\to\infty} \sum_{i=0}^{n-1} F'(t_i)\,\Delta t \qquad (*)$$

Remember: $\Delta t = \frac{b-a}{n}$. First let me show you the conceptual idea of the proof (or, how a physicist would "prove" it.) Then for those interested, I will give the mathematically rigorous proof.

The definition of the derivative tells us that

$$F'(t_i) = \lim_{\Delta t\to 0} \frac{F(t_i+\Delta t) - F(t_i)}{\Delta t}.$$

So, if we choose Δt to be very small (equivalently, n to be very large,) then we have

approximately equal

$$F'(t_i) \approx \frac{F(t_i+\Delta t) - F(t_i)}{\Delta t}$$

Notice that $t_i + \Delta t = t_{i+1}$. (think about the picture,) so

$$F'(t_i) \approx \frac{F(t_{i+1}) - F(t_i)}{\Delta t}.$$

Going back to $(*)$, supposing n is chosen large, we have

$$\int_a^b F'(t)\,dt \approx \sum_{i=0}^{n-1} \left(\frac{F(t_{i+1}) - F(t_i)}{\Delta t}\right)\Delta t = \sum_{i=0}^{n-1} F(t_{i+1}) - F(t_i).$$

Here is the sneaky part: let's write out this last sum, I'll list the terms in descending order for clarity.

$$\sum_{i=0}^{n-1} F(t_{i+1}) - F(t_i) = \left(F(t_n) - F(t_{n-1})\right) + \left(F(t_{n-1}) - F(t_{n-2})\right) + \left(F(t_{n-2}) - \cdots\right.$$

$$\cdots + \left(F(t_2) - F(t_1)\right) + \left(F(t_1) - F(t_0)\right).$$

Check it out: all the terms in the middle cancel out!

This leaves us with:

$$\sum_{i=0}^{n-1} F(t_{i+1}) - F(t_i) = F(t_n) - F(t_0).$$ Finally we realize that $t_n = b$ and $t_0 = a$.

Putting it all together gives

$$\int_a^b F'(t)\,dt = F(b) - F(a). \quad \blacksquare$$

The insight of the proof (what really makes this work) is noticing all that cancellation in the sum. Sums like this are called "telescopic."

We fudged a bit with the limits in our proof because the definition of a limit is technical. Let me go back and fill in the details.

We want to show that $\lim_{n \to \infty} \sum_{i=0}^{n-1} F'(t_i)\Delta t = F(b) - F(a)$.

The definition of limit says we need to show that for every positive number $\varepsilon > 0$, we can find some large N so that for any $n \geq N$, we have $\left| \sum_{i=0}^{n-1} F'(t_i)\Delta t - \left(F(b) - F(a)\right)\right| < \varepsilon$.

(Do you see why this is saying the value of the sum is getting closer to $F(b)-F(a)$ for large values of n?)

So let $\varepsilon > 0$ be any positive number and let's find N. For each t_i, we use the definition of the derivative to say there is some $\delta > 0$ so that whenever $0 < \Delta t < \delta$, we have

$$\left| F'(t_i) - \left(\frac{F(t_i + \Delta t) - F(t_i)}{\Delta t} \right) \right| < \frac{\varepsilon}{b-a}$$ (← you will see where this comes from in a minute.)

so

$$\left| F'(t_i)\Delta t - \left(F(t_i + \Delta t) - F(t_i) \right) \right| < \frac{\varepsilon}{b-a} \Delta t = \frac{\varepsilon}{n}$$ (because $\Delta t = \frac{b-a}{n}$.)

There is a possibility that δ depends on t_i which gives us a headache because we want one δ to work for all the t_i.

I need to assume $F'(t)$ continuous.

{Here I cite a theorem which tells us that because $[a,b]$ is a <u>closed</u> interval of <u>finite length</u>, we can find one δ that works, (we say F is "uniformly differentiable" on $[a,b]$.)

Okay, now let N be any number $> \frac{b-a}{\delta}$, we need to show this works.

If $n \geq N > \frac{b-a}{\delta}$, then $\delta > \frac{b-a}{n} = \Delta t$, so for each i we have

$$\left| F'(t_i)\Delta t - \left(F(t_{i+1}) - F(t_i) \right) \right| < \frac{\varepsilon}{n}.$$

Therefore, $$\left| \sum_{i=0}^{n-1} F'(t_i)\Delta t - \left(F(t_{i+1}) - F(t_i) \right) \right|$$

This step uses the "triangle inequality":

$$\leq \sum_{i=0}^{n-1} \left| F'(t_i)\Delta t - \left(F(t_{i+1}) - F(t_i) \right) \right|$$

$$< \sum_{i=0}^{n-1} \frac{\varepsilon}{n} = \varepsilon$$

Finally, note that

$$\left| \sum_{i=0}^{n-1} F'(t_i)\Delta t - \left(F(t_{i+1}) - F(t_i)\right) \right|$$

$$= \left| \sum_{i=0}^{n-1} F'(t_i)\Delta t - \sum_{i=0}^{n-1} F(t_{i+1}) - F(t_i) \right|$$

cancellation trick!

$$= \left| \sum_{i=0}^{n-1} F'(t_i)\Delta t - \left(F(b) - F(a)\right) \right|,$$

So, we have shown that for any $\varepsilon > 0$, choosing any $N > \frac{b-a}{\delta}$ gives us, for any $n \geq N$,

$$\left| \sum_{i=0}^{n-1} F'(t_i)\Delta t - \left(F(b) - F(a)\right) \right| < \varepsilon,$$

which, by the definition of limit means that

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} F'(t_i)\Delta t = F(b) - F(a),$$ which is what we wanted. ∎

Phew, that was a lot more work than I had anticipated. Kudos if you're still following.