Euler's Formula,

The founders of calculus understood functions in an incredibly deep way that comes from years of using, computing, and thinking about them. Our modern pedagological approach is historically completely backwards; we finish our courses where they began, which was with power series. One of their best tricks was to think of smooth functions as <u>power series</u> which is like a polynomial that goes on forever.

For example, the morally correct way to define the function e^x is by its power series:

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \cdots = \sum_{n \geq 0} \frac{x^n}{n!}.$$

Then we define the number e to be what we get when we plug in $x = 1$:

$$e = e^1 = \sum_{n \geq 0} \frac{1}{n!}$$ (there are other definitions which came first.)

Given a function $f(x)$ which is smooth (has all of its derivatives) we can find a "power series expansion" also called a "Taylor series expansion" by the (kind of scary) formula

$$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(c)}{n!}(x-c)^n$$ where c is any number we want, called the "center." $f^{(n)}(c)$ means "n^{th} derivative of f evaluated at $x = c$."

Let's not get too sidetracked with this formula for Taylor series; it's cool, but it's a long story. The point is, if we start with our favorite function and some center c, we can cook up the Taylor series for the function centered at $x=c$.

For example, let $c=0$, then

You can derive these!

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots = \sum_{n\geq 0} \frac{(-1)^n x^{2n}}{(2n)!}$$

Ready? Here is Euler's great insight:

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{6} + \frac{(ix)^4}{24} + \frac{(ix)^5}{120} + \frac{(ix)^6}{720} + \cdots$$

(powers of i repeat with period 4)

$$= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{6} + \frac{x^4}{24} + i\frac{x^5}{120} - \frac{x^6}{720} + \cdots$$

(collect real and imaginary parts)

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$
$$+ i\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)$$

$$= \cos x + i \sin x$$

So, $e^{ix} = \cos x + i \sin x$! Woah.

<u>Remark:</u> The naive definition of exponentiation a^b tells us to multiply together b copies of a. This gets confusing when b is not a whole number and profoundly mystifying when b is not a real number. This is one benefit of using the power series definition for e^x, since it only requires us to do addition and multiplication with x, which we know how to do.

Things to try:

1) Compute derivatives of both sides of $e^{ix} = \cos x + i \sin x$, are the answers consistent?

2) Plug-in nice values for x (think: angles where we can compute $\cos x$ and $\sin x$) to derive famous identities. Hint: try $x = \pi$.

3) If a, b are real numbers, then $a + bi$ is a complex number. The real numbers live on a number line, the complex numbers live on a number plane. Think: $a + bi \longmapsto (a, b)$
With this in mind, where is the point e^{ix} on the complex plane? To reduce cognitive dissonance, maybe write this as $e^{i\theta}$. Can you see why mathematicians write complex numbers as $re^{i\theta}$ and call this "polar form"?

4) Follow-up: If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are two complex numbers, then we can multiply them to get

$$z_1 z_2 = (r_1 r_2)\, e^{i(\theta_1 + \theta_2)}.$$

Can you use this to give a geometric interpretation of multiplication of complex numbers?

5) Follow-Follow-up: When we were young, our mother's told us that $(-1)(-1) = 1$ but they never told us why.
Writing -1 in polar form it is $-1 = e^{i\pi}$. Can you use 4) to explain why $(-1)(-1) = 1$ is reasonable?