Math 116 — Practice for Exam 3

Generated November 30, 2017

Name: **SOLUTIONS**

Instructor: _____ Section Number: _____

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1. This exam has 6 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2014	2	3	potion	16	
Fall 2015	3	3	rumor	13	
Winter 2015	2	6	caffeine drip	9	
Fall 2015	2	7	currency	18	
Winter 2017	2	3	leaky boat	5	
Fall 2016	2	3	election	5	
Total				66	

Recommended time (based on points): 64 minutes

3. [16 points]

a. [5 points] Kazilla uses the ingredients you acquired from the store to make a special potion. Kazilla starts with a bucket containing 5 L of water and steadily adds purple liquid to the bucket at a rate of 0.5 L/min. Once in the bucket, the purple liquid mixes instantly with the water and the mixture drains out of the bottom of the bucket at a rate of 0.5 L/min. Let $P(t)$ give the amount of purple liquid in the bucket t minutes after Kazilla starts making the potion. Write a differential equation involving $P(t)$.

Solution:

$$\frac{dP}{dt} = 0.5 - \frac{P}{10}$$

b. [7 points] Later, Kazilla asks you to find the correct amount of green liquid to add to her potion. Let $G(t)$ be the total amount of green liquid you need to add, in liters, after t minutes. Suppose $G(1) = \frac{3}{2}$ and $G(t)$ satisfies the differential equation

$$\frac{dG}{dt} = -2t(G-1)^2.$$

Find $G(t)$.

Solution: Start with a separation of variables:

$$\int \frac{dG}{(G-1)^2} = \int -2t\,dt$$

$$\frac{-1}{(G-1)} = -t^2 + C$$

$$G = \frac{1}{t^2 + C} + 1$$

Using the initial condition that $G(1) = 3/2$, we can solve for C:

$$\frac{3}{2} = \frac{1}{1+C} + 1$$

$$C = 1$$

So $G(t) = \dfrac{1}{t^2+1} + 1$

c. [4 points] Finally, Kazilla asks you to add an amount of blue liquid $B(t)$ at a rate given by

$$\frac{dB}{dt} = -2B + e^{t/2}.$$

For which value(s) of c is the function

$$B(t) = 7e^{-2t} + \frac{ce^{t/2}}{5}$$

a solution to this differential equation?

Solution:

$$B'(t) = -14e^{-2t} + \frac{ce^{t/2}}{10}$$

$$B'(t) = \frac{dB}{dt} = -2(7e^{-2t} + \frac{ce^{t/2}}{5}) + e^{t/2} = -14e^{-2t} + (1 - \frac{2c}{5})e^{t/2}$$

So $1 - \frac{2c}{5} = \frac{c}{10}$ or $c = 2$.

3. [13 points]

a. [4 points] The number of people R that have heard a rumor increases at a rate proportional to the product of the number of people that have heard the rumor and the number of people that haven't yet heard the rumor. Write a differential equation for R which models the scenario described assuming that the total number of people is 1,000. Use $k > 0$ for the constant of proportionality.

Solution: The number of people that have heard the rumor is R, so the number of people that have not yet heard the rumor is $1000 - R$. Thus the differential equation is $\frac{dR}{dt} = kR(1000 - R)$.

$$\frac{dR}{dt} = \underline{\qquad kR(1000 - R) \qquad}$$

b. [4 points] For what values of A, B is $y(t) = At\cos t + Bt$ a solution to the differential equation $ty' = y + t^2 \sin t$ satisfying the initial condition $y\left(\frac{\pi}{2}\right) = 2\pi$? Be sure to show your work.

Solution: Since $y'(t) = A\cos t - At\sin t + B$, $y(t)$ is a solution if

$$t(A\cos t - At\sin t + B) = At\cos t + Bt + t^2\sin t \qquad \Rightarrow \qquad -At^2\sin t = t^2\sin t$$

Thus $A = -1$. Plugging in the initial condition $y\left(\frac{\pi}{2}\right) = 2\pi$,

$$-\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + B\left(\frac{\pi}{2}\right) = 2\pi.$$

Thus $B = 4$.

$A =$ **-1**

$B =$ **4**

c. [5 points] Find the solution to the differential equation

$$e^{-x} + y^2\frac{dy}{dx} = 0, \quad \text{with initial condition} \quad y(0) = 2.$$

Solution: Moving the e^{-x} to the right side of the equation and separating variables,

$$\int y^2 dy = \int -e^{-x} dx$$
$$\frac{1}{3}y^3 = e^{-x} + C$$
$$y = \sqrt[3]{3e^{-x} + C}$$

Plugging in the initial condition $y(0) = 2$, $2 = \sqrt[3]{3 + C}$. Therefore $C = 5$.

$$y = \underline{\qquad \sqrt[3]{3e^{-x} + 5} \qquad}$$

6. [9 points] An extremely sleepy graduate student is grading Math 116 exams. She has been drinking coffee all day, but it just is not enough. She hooks up a caffeine drip that delivers caffeine into her body at a constant rate of 170 mg/hr. The amount of caffeine in her body decays at a rate proportional to the current amount of caffeine in her body. The half-life of caffeine in her body is 6 hours.

a. [4 points] Using the blank provided, write a differential equation which models the scenario described above. Use $Q(t)$ for the amount of caffeine in the graduate student's body, measured in mg, t for hours after she hooked up the caffeine drip, and $k > 0$ for the constant of proportionality.

Solution: The rate that the amount of caffeine in the graduate student's body is changing over time should be the rate that caffeine is entering their body minus the rate that caffeine is leaving their body. The rate that caffeine is entering the graduate students body is a constant 170 mg/hr. The rate that caffeine is leaving the graders body is proportional to the current amount, so it is kQ mg/hr. Putting all this together gives us the equation written below.

$$\frac{dQ}{dt} = \underline{\qquad 170 - kQ \qquad}$$

b. [5 points] Use the half-life of caffeine to determine the constant of proportionality.

Solution: We know that the amount of caffeine in the graduate student's body decays exponentially with decay rate k. If C_0 is the initial amount of caffeine, then a half-life of 6 hours means that

$$\frac{1}{2}C_0 = C_0 e^{-k6}.$$

Solving for k gives us that $k = -\frac{1}{6}\ln\left(\frac{1}{2}\right)$.

7. [18 points] A certain small country called Merrimead has 25 million dollars in paper currency in circulation, and each day 50 thousand dollars comes into Merrimead's banks. The government decides to introduce new currency by having the banks replace the old bills with new ones whenever old currency comes into the banks. Assume that the new bills are equally distributed throughout all paper currency. Let $M = M(t)$ denote the amount of new currency, in thousands of dollars, in circulation at time t days after starting to replace the paper currency.

a. [5 points] Write a differential equation involving $M(t)$, including an appropriate initial condition.

Solution: The concentration of old bills among all bills in circulation is $\frac{25000-M}{25000}$, and 50 thousand dollars moves through the bank each day, so

$$\frac{dM}{dt} = \text{Concentration} \times \text{Money per day} = \frac{25000 - M}{25000} \cdot 50, \qquad M(0) = 0.$$

Now consider the differential equation

$$B^2 + 2B\frac{dB}{dt} = 2500.$$

b. [4 points] Find all equilibrium solutions and classify their stability.

Solution: If $\frac{dB}{dt} = 0$ then we see that $B^2 = 2500$, so the equilibrium solutions are $B = \pm 50$. Both equilibrium solutions are stable.

Brightcrest, a second small country, also wants to replace all of their old paper bills as well, using a different strategy than Merrimead. The amount $B(t)$, in millions of dollars, of new paper currency in circulation in Brightcrest at a time t years after starting to replace the paper currency is modeled by the differential equation for B above with initial condition $B(0) = 0$.

c. [6 points] Find a formula for $B(t)$.

Solution: Using separation of variables, we have $\int \frac{2B}{2500-B^2}\, dB = \int dt$. Using the substitution $w = 2500 - B^2, dw = -2B\, dB$, we see that $-\ln|2500 - B^2| = t + C$. Solving for B, we see $B = \sqrt{2500 - Ae^{-t}}$.
Since $B(0) = 0$, we see that $A = 2500$, so

$$B(t) = \sqrt{2500 - 2500e^{-t}}.$$

d. [3 points] Assuming that all of the old bills are replaced in the long run, how much time will pass after starting to replace the paper bills until the new currency accounts for 99% of all currency in Brightcrest?

Solution: Since all of the bills are replaced in the long run, the total amount of money in circulation is $\lim_{t\to\infty} B(t) = 50$ million dollars. So the amount of time that passes until the new bills account for 99% of all currency is the value of t so that

$$(.99)(50) = \sqrt{2500 - 2500e^{-t}}.$$

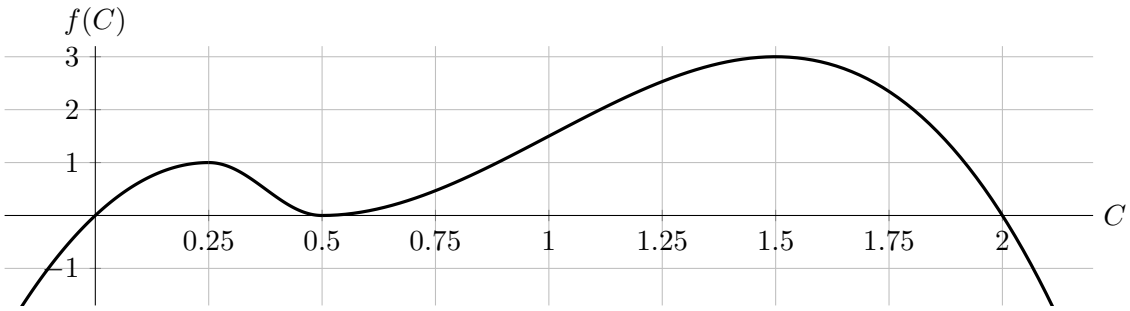
So $t = -\ln(1 - (.99)^2) \approx 3.92$ years.

3. [5 points] Sasha and her friends are sipping lemonade on her boat when the boat begins to leak through a new hole in the bottom. Water begins to enter the boat at a constant rate of 1.5 gallons per minute. Immediately, they spring into action and begin to scoop the water out of the boat using lemonade pitchers that hold 0.25 gallons of water. That rate that the water is scooped, water, in scoops per minute, is proportional to the cube root of the volume of water currently in the boat, with constant of proportionality k. Let $W = W(t)$ be the volume of water in the boat, in gallons, t minutes after the leak begins. Write a differential equation that models $W(t)$, and give an appropriate initial condition.

Answer: Differential Equation: $\frac{dW}{dt} = 1.5 - 0.25kW^{1/3}$

Initial Condition: $W(0) = 0$

4. [6 points] Consider the differential equation $\frac{dC}{dt} = f(C)$ where $f(C)$ is the function graphed below.



a. [4 points] Identify all equilibrium solutions of this differential equation. Then indicate which of these equilibrium solutions are stable. Write your answers on the answer blanks provided.

Answer: All Equilibrium Solutions: $C = 0,\ C = 0.5,\ C = 2$

Stable Equilibrium Solutions: $C = 2$

b. [2 points] Suppose that a solution to this differential equation passes through a point with $C = 0.17$. For this solution, what will happen to the value of C as $t \to \infty$?

Solution: Note that $\frac{dC}{dt} > 0$ for $0.17 \le C < 0.5$, and that $C = 0.5$ is an equilibrium solution. So C will increase from 0.17 and approach 0.5. That is, $\lim_{t\to\infty} C = 0.5$.

2. [5 points] Find constants A and B so that the function $h(w)$, defined for $w > 0$ by

$$h(w) = Aw^3 + \frac{1}{w},$$

is a solution to the differential equation

$$w^2 \frac{dh}{dw} - 3wh + B = 0$$

satisfying $h(1) = \frac{3}{2}$. Show all your work, and write your final answers in the spaces provided.

Solution:

$A =$ 1/2

$B =$ 4

3. [5 points] In a recent presidential election between candidate A and candidate B, Shamcorp's rival company Hawk-I tried fixing the election by changing the votes on some of the ballots. For the last three hours of the election (between 5pm and 8pm), the company gained access to the huge ballot box containing 100 million ballots.

Hawk-I employees removed ballots from the ballot box continuously at a rate of 4 million ballots per hour. Those ballots were removed in proportion to the current ratio in the box. Hawk-I employees then instantly changed the the ballots voting for candidate B to vote for candidate A (leaving any votes for candidate A unchanged) before immediately returning the ballots to the box.

Assume that the ballot box always contains 100 million votes, and that the ballot box only contains votes for candidates A and B.

Write a differential equation that models $a(t)$, the number of ballots voting for candidate A, **in millions**, in the ballot box t hours after Hawk-I began changing votes.

Solution:

$$\frac{da}{dt} = 4 - \frac{a}{25}$$