Lecture Notes on Game Theory

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1. Definitions

Definition 1. A strategic game is a list \((N, (A_i)_{i\in N}, (u_i)_{i\in N})\) where \(N\) is a finite set of the form \(N = \{1, 2, \ldots, n\}\) (with \(n \geq 2\)), for every \(i \in N\) \(A_i\) is a non-empty, finite set, and for every \(i \in N\) \(u_i\) is a real-valued function with domain \(A = \times_{i\in N} A_i\).

\(N\) is the set of players. \(A_i\) is the set of player \(i\)’s actions, and \(u_i\) is player \(i\)’s von Neumann Morgenstern utility function. We write \(a_i\) for elements of \(A_i\). We refer to elements of \(A_i\) also as pure (as opposed to mixed) actions of player \(i\). We write \(a\) for elements of \(A\). We write \(A_{-i}\) for \(\times_{j\in N, j\neq i} A_j\). We write \(a_{-i}\) for elements of \(A_{-i}\).

Definition 2. A mixed action \(\alpha_i\) for player \(i\) is a probability distribution over \(A_i\).

We identify the mixed action \(\alpha_i\) that place probability 1 on action \(a_i\) with this action itself. For any finite set \(X\) we write \(\Delta(X)\) for the set of probability distributions over \(X\), and thus \(\Delta(A_i)\) is the set of mixed actions of player \(i\). We write \(\alpha\) for elements of \(\times_{i\in N} \Delta(A_i)\). We write \(\alpha_{-i}\) for elements of \(\times_{j\in N, j\neq i} \Delta(A_j)\). The expected utility of player \(i\) when players choose mixed action profile \(\alpha\) is:

\[
U_i(\alpha) = \sum_{(a_1, a_2, \ldots, a_n) \in A} \left( u_i(a_1, a_2, \ldots, a_n) \prod_{j \in N} \alpha_j(a_j) \right)
\]

Note that this definition implicitly assumes that different players’ randomizations when implementing their mixed action are independent. We shall make this assumption throughout.

2. Best Replies and Actions that are not Strictly Dominated

Two basic ideas about rationality in games are (i) rational players only choose actions that maximize expected utility for some belief about the other players’ actions, and (ii) rational players only choose actions that are not strictly dominated by other actions. Here, we formalize these two ideas,
and then demonstrate that they imply exactly the same predictions about the behavior of rational players.

**Definition 3.** A belief $\mu_i$ of player $i$ is an element of $\Delta(A_{-i})$. An action $a_i \in A_i$ is a best reply to a belief $\mu_i$ if

$$a_i \in \arg \max_{a'_i \in A_i} \sum_{a_{-i} \in A_{-i}} \left( u_i(a'_i, a_{-i}) \mu_i(a_{-i}) \right).$$

Note that we do not require beliefs to be the products of their marginals, that is, we don’t require player $i$ to believe that the other players’ actions are stochastically independent. If we did, Proposition 1 below would not be true.

**Definition 4.** An action $a_i \in A_i$ is strictly dominated by a mixed action $\alpha_i \in \Delta(A_i)$ if

$$U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i}) \quad \forall a_{-i} \in A_{-i}.$$  

Note that we allow the strictly dominating action to be a mixed action. If we did not, then Proposition 1 below would not be true.

**Proposition 1.** An action $a_i^* \in A_i$ is a best reply to some belief $\mu_i \in \Delta(A_{-i})$ if and only if $a_i^*$ is not strictly dominated.

**Proof.** Step 1: We prove the “only if” part, that is, we assume that $a_i^*$ is a best reply to a belief $\mu_i \in \Delta(A_{-i})$, and infer that $a_i^*$ is not strictly dominated. The proof is indirect. Suppose $a_i^*$ were strictly dominated by $\alpha_i \in \Delta(A_i)$. Then, obviously, $\alpha_i$ yields strictly higher expected utility given the belief $\mu_i$ than $a_i^*$:

$$\sum_{a_{-i} \in A_{-i}} U_i(\alpha_i, a_{-i}) \mu_i(a_{-i}) > \sum_{a_{-i} \in A_{-i}} u_i(a_i^*, a_{-i}) \mu_i(a_{-i}).$$  

We thus have that $\alpha_i$ is a better response to $\mu_i$ than $a_i^*$, which is almost what we want to obtain, but not quite. To obtain the desired contradiction, we want to find a pure action that is a better response to $\mu_i$ than $a_i^*$. This can be done as follows. We re-write the left hand side of (4) as follows:

$$\sum_{a_{-i} \in A_{-i}} U_i(\alpha_i, a_{-i}) \mu_i(a_{-i}) = \sum_{a_{-i} \in A_{-i}} \sum_{a_i \in A_i} \alpha_i(a_i) u_i(\alpha_i, a_{-i}) \mu_i(a_{-i})$$

$$= \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \alpha_i(a_i) u_i(\alpha_i, a_{-i}) \mu_i(a_{-i})$$

$$= \sum_{a_i \in A_i} \alpha_i(a_i) \left( \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \mu_i(a_{-i}) \right).$$
Combining (4) and (5) we have:

\[ \sum_{a_i \in A_i} \alpha_i(a_i) \left( \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \mu_i(a_{-i}) \right) > \sum_{a_{-i} \in A_{-i}} u_i(a_i^*, a_{-i}) \mu_i(a_{-i}). \]

The left hand side of (6) is a convex combination of the expressions in large brackets in that term. This convex combination can be larger than the right hand side of (6) only if one of the expressions in large brackets is larger than the right hand side of (6), i.e., for some \( a_i \in A_i \):

\[ \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \mu_i(a_{-i}) > \sum_{a_{-i} \in A_{-i}} u_i(a_i^*, a_{-i}) \mu_i(a_{-i}). \]

and thus \( a_i \) is a better response to \( \mu_i \) than \( a_i^* \), which contradicts the assumption \( a_i^* \) is a best response to \( \mu_i \) among all pure actions.

**Step 2.** We prove the “if” part, that is, we assume that \( a_i^* \) is not strictly dominated, and we show that there is a belief \( \mu_i \in \Delta(A_{-i}) \) to which \( a_i^* \) is a best response. The proof is constructive. We define two subsets, \( X \) and \( Y \), of the set \( \mathbb{R}^{|A_{-i}|} \), that is, the Euclidean space with dimension equal to the number of elements of \( A_{-i} \). We shall think of the elements of these sets as payoff vectors. Each component indicates a payoff that player \( i \) receives when the other players choose some particular \( a_{-i} \in A_{-i} \).

Now pick any one-to-one mapping \( f : A_i \rightarrow \{1, 2, \ldots, |A_{-i}|\} \). For any action \( a_i \) of player \( i \), we write \( u_i(a_i, \langle a_{-i} \rangle) \in \mathbb{R}^{|A_{-i}|} \) for the vector of payoffs that player \( i \) receives when playing \( a_i \), and when the other players play their various action combinations \( a_{-i} \). Specifically, the \( k \)-th entry of \( u_i(a_i, \langle a_{-i} \rangle) \) is the payoff \( u_i(a_i, f^{-1}(k)) \) where \( f^{-1} \) is the inverse of \( f \). Intuitively, \( f \) defines an order in which we enumerate the elements of \( A_{-i} \), and \( u_i(a_i, \langle a_{-i} \rangle) \) lists the payoffs of player \( i \) when he plays \( a_i \) and the other players play \( a_{-i} \) in the order defined by \( f \).

The set \( X \) is:

\[ X = \{ x \in \mathbb{R}^{|A_{-i}|} | x > u_i(a_i^*, \langle a_{-i} \rangle) \}. \]

Here, “\( > \)” is to be interpreted as: “strictly greater in every component.” Therefore, the set \( X \) is the set of payoff vectors that are strictly greater in every component than \( u_i(a_i^*, \langle a_{-i} \rangle) \), that is the payoff vector that corresponds to the undominated action \( a_i^* \).

The set \( Y \) is:

\[ Y = co \{ y \in \mathbb{R}^{|A_{-i}|} | \exists a_i \in A_i : y = u_i(a_i, \langle a_{-i} \rangle) \}. \]

Here, “\( co \)” stands for “convex hull.” The payoff vectors in \( Y \) are the payoff vectors that player \( i \) can achieve by choosing some mixed action. The weight that the convex combination that defines an element of \( y \) places on each place was after the end of the proof.

\[ \text{An example and a graph that illustrate Step 2 follow after the end of the proof.} \]
element of \( \{ x \in \mathbb{R}^{A_{-i}} \mid \exists a_i \in A_i : x = u_i(a_i, (a_{-i})) \} \) corresponds to the probability which the mixed action assigns to each pure action \( a_i \in A_i \).

It is obvious that both sets are nonempty and convex sets. Moreover, their intersection is empty. If \( X \) and \( Y \) overlapped, then every common element would correspond to the payoffs arising from a mixed action of player \( i \) that strictly dominates \( a^*_i \). Because by assumption no such mixed action exists, \( X \) and \( Y \) cannot have any elements in common.

In the previous paragraph we have checked all the assumptions of the separating hyperplane theorem: we have two nonempty and convex sets that have no elements in common. The separating hyperplane theorem (Theorem 1.68 in Sundaram [4]) then says that there exist some row vector \( \pi \in \mathbb{R}^{A_{-i}} \) which is not equal to zero in every component, and some \( \delta \in \mathbb{R} \), such that:

\[
\pi \cdot x \geq \delta \quad \forall x \in X
\]

and

\[
\pi \cdot y \leq \delta \quad \forall y \in Y.
\]

Here \( \cdot \) stands for the scalar product of two vectors. We treat all vectors in \( X \) and \( Y \) as column vectors. Therefore, the above scalar products are well-defined.

We now make two observations. The first is:

\[
\delta = \pi \cdot u_i(a^*_i, (a_{-i})).
\]

To show this note that by definition \( u_i(a^*_i, (a_{-i})) \in Y \), and therefore, by (11), \( \pi \cdot u_i(a^*_i, (a_{-i})) \leq \delta \). Next, for every \( n \in \mathbb{N} \) define \( x_n = u_i(a^*_i, (a_{-i})) + e^n \cdot \iota \), where \( e \in (0, 1) \) is some constant and \( \iota \) is the column vector in \( \mathbb{R}^{A_{-i}} \) in which all entries are “1”. Observe that for every \( n \in \mathbb{N} \) we have \( x_n \in X \), so that by (10) we have: \( \pi \cdot x_n \geq \delta \) for every \( n \). On the other hand, we have: \( \lim_{n \to \infty} x_n = u_i(a^*_i, (a_{-i})) \). By the continuity of the scalar product of vectors therefore: \( \pi \cdot u_i(a^*_i, (a_{-i})) \geq \delta \). This, combined with our earlier observation \( \pi \cdot u_i(a^*_i, (a_{-i})) \leq \delta \) implies (12).

Our second observation is:

\[
\pi \geq 0
\]

where we interpret \( \geq \) to mean “greater or equal in every component,” but not identical, and 0 stands for the vector consisting of zeros only. That \( \pi \) is not equal to 0 is part of the assertion of the separating hyperplane theorem. We shall prove indirectly that no component of \( \pi \) can be negative. Assume that \( \pi \) has a negative component. Without loss of generality, assume that it is the first. Now we consider the vector \( x = u_i(a^*_i, (a_{-i})) + (1, e, e, \ldots, e) \) where \( e > 0 \) is some number. Clearly, the vector \( x \) that we define like this is contained in \( X \). Moreover,

\[
\pi \cdot x = \pi \cdot u_i(a^*_i, (a_{-i})) + \pi \cdot (1, e, e, \ldots, e) = \delta + \pi \cdot (1, e, e, \ldots, e)
\]
where for the second equality we have used (12). We now want to evaluate \( \pi \cdot (1, \varepsilon, \varepsilon, \ldots, \varepsilon) \). Unfortunately, we need some additional notation: we shall use for the \( k \) -th component of the vector \( \pi \) the symbol: \( \pi_k \). Here, \( k \in \{1, 2, \ldots, |A_{-i}|\} \). We then obtain:

\[
\pi \cdot (1, \varepsilon, \varepsilon, \ldots, \varepsilon) = \pi_1 + \varepsilon \sum_{k=2}^{|A_{-i}|} \pi_k
\]

Now observe that by the contrapositive assumption of our indirect proof \( \pi_1 < 0 \). Therefore, for small enough \( \varepsilon \) the right hand side of (15) is negative. Using this fact, we obtain from (14):

\[
\pi \cdot x < \delta
\]

which contradicts \( x \in X \) and (10). This completes our indirect proof of (13).

Now we denote by \( ||\pi|| \) the Euclidean norm of \( \pi \). Because \( \pi \neq 0 \), if we define \( \mu_i \) by:

\[
\mu_i = \frac{1}{||\pi||} \pi,
\]

then the Euclidean norm of \( \mu_i \) is 1, and thus \( \mu_i \in \Delta(A_{-i}) \), that is, \( \mu_i \) is a belief of player \( i \). We complete the proof by showing that \( a^*_i \) is a best response to \( \mu_i \). (11) and (12) together imply:

\[
\pi \cdot y \leq \pi \cdot u_i(a^*_i, \langle a_{-i} \rangle) \quad \forall y \in Y.
\]

Dividing this inequality by \( ||\pi|| \) we get:

\[
\mu_i \cdot y \leq \mu_i \cdot u_i(a^*_i, \langle a_{-i} \rangle) \quad \forall y \in Y.
\]

Now by definition of \( Y \) for every \( a_i : u_i(a_i, \langle a_{-i} \rangle) \in Y \). Therefore:

\[
\mu_i \cdot u_i(a_i, \langle a_{-i} \rangle) \leq \mu_i \cdot u_i(a^*_i, \langle a_{-i} \rangle) \quad \forall a_i \in A_i.
\]

This is what we wanted to prove: \( a^*_i \) is a best response in \( A_i \) to \( \mu_i \). \( \square \)

We illustrate Step 2 of the proof of Proposition 1 with the following game in which player 1 chooses rows and player 2 chooses columns. Only the payoffs of player 1 are shown.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>MA</td>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>MB</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>MC</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>-2</td>
<td>3</td>
</tr>
</tbody>
</table>

Example 1
Player 1’s action $T$ is not strictly dominated. We illustrate in Figure 1 the construction of beliefs to which action $T$ is a best reply. The sets $X$ and $Y$ to which Step 2 of the proof of Proposition 1 refers are shown in the figure.

![Figure 1](image)

The hyperplane (straight line) separating $X$ and $Y$ is the dashed line in Figure 1. Figure 1 also shows the orthogonal vector for this hyperplane.

3. Cautious Best Replies and Actions that are not Weakly Dominated

A cautious player might hold a belief that attaches some strictly positive, although possibly arbitrarily small, probability to every action profile of the other players. Here we show that the set of best responses to such beliefs is exactly the set of actions that are not weakly dominated.

**Definition 5.** An action $a_i \in A_i$ is weakly dominated by a mixed action $\alpha_i \in \Delta(A_i)$ if

$$U_i(\alpha_i, a_{-i}) \geq u_i(a_i, a_{-i}) \quad \forall a_{-i} \in A_{-i}$$

and

$$U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i}) \quad \exists a_{-i} \in A_{-i}.$$  

**Proposition 2.** In a finite strategic game an action $a_i^*$ is a best reply among all pure actions of player $i$ to a belief $\mu_i \in \Delta(A_{-i})$ with full support, i.e. $\mu_i(a_{-i}) > 0$ for every $a_{-i} \in A_{-i}$, if and only if $a_i^*$ is not weakly dominated.
Before proving Proposition 2 we shall illustrate by means of an example that the construction of Step 2 of the proof of Proposition 1 does not necessarily yield a belief $\mu_i$ with full support and therefore can not be used to prove Proposition 1. Indeed, that construction can be done even if an action is weakly dominated. In this case the construction yields a belief that assigns zero probability to some of the action combinations of the other players. Our example is the same as example 1 except that we assume that player 1 has an additional action that yields payoffs 3 and 2, depending whether player 2 chooses L or R. This action thus weakly dominates T. We also assume that action B yields payoffs -2 and 2 rather than -2 and 3, as was the case before. This is to avoid that a mixed action strictly dominates $T$. The sets $X$ and $Y$ from Step 2 of the proof of Proposition 1 for this modified example are shown in Figure 2.

![Figure 2](image)

The hyperplane separating $X$ and $Y$ is the dashed line in Figure 2. The orthogonal vector that we have drawn has coordinates 0 and 1, which correspond to the belief of player 1 that puts probability 1 on player 2 playing R. The construction in the proof of Proposition 1 allows thus beliefs that put probability zero on some actions. We demonstrate below how a modified construction shows that full support beliefs can be obtained whenever an action is not weakly dominated.

*Proof.* The argument that proves that a best response to a full support belief is not weakly dominated is essentially the same as Step 1 of the proof of Proposition 1, and we omit it.\(^3\)

\(^3\)All that changes is the reasoning that leads to inequality (4), and which we did not even spell out in the proof of Proposition 1.
The proof of the converse is a less obvious modification of the argument in Step 2 of the proof of Proposition 1. As in that proof, we shall construct a the belief to which a not weakly dominated action is a best reply, and that belief will be the orthogonal vector of a hyperplane separating two sets $X$ and $Y$. To ensure that this belief has full support, we shall use a stronger separating hyperplane theorem than we used in the proof of Proposition 1. Specifically, it will be useful to appeal to a separating hyperplane theorem that allows us to replace the weak inequality in (10) above by a strict inequality. Such a theorem is the “strong separating hyperplane theorem ” which is Theorem 6 in Border [2]. To apply this theorem we need, in particular, that the set $X$ is compact. To achieve this, we have to change the definition of $X$ substantially. This will be done below. This change in the definition of $X$ will then also necessitate a small change in the definition of $Y$. Hopefully, the remainder of the proof will clarify how these changes in the definitions of $X$ and $Y$ are used in the proof.

We define $X$ and $Y$ as follows:

\[(23) \quad X = \{x \in \mathbb{R}^{|A-i|} \mid x = u_i(a^*_i, \langle a_{-i} \rangle) + y \text{ for some } y \in \Delta \}.\]

(where $\Delta$ is the unit simplex in $\mathbb{R}^{|A-i|}$) and

\[(24) \quad Y = \left\{ y \in \mathbb{R}^{|A-i|} \mid y = u_i(a^*_i, \langle a_{-i} \rangle) + \sum_{a_i \in A_i} \lambda(a_i) (u_i(a_i, \langle a_{-i} \rangle) - u_i(a^*_i, \langle a_{-i} \rangle)) \right\}.\]

Thus, $X$ is now the set of all payoff vectors that result if we add an element of the unit simplex to $u_i(a^*_i, \langle a_{-i} \rangle)$, whereas before it was the set of all payoff vectors that are strictly greater in every component than $u_i(a^*_i, \langle a_{-i} \rangle)$. Note that the modified definition of $X$ leaves $X$ a convex set. Obviously, $X$ is also compact. The set $Y$ is the smallest cone with vertex $u_i(a^*_i, \langle a_{-i} \rangle)$ that includes the set $Y$ as previously defined. Observe that with this new definition $Y$ is convex. By Lemma 6 in Border [3] it is also closed because it is a finitely generated convex cone. It is generated by the finite collection of vectors $u_i(a_i, \langle a_{-i} \rangle) - u_i(a^*_i, \langle a_{-i} \rangle)$ (where $a_i \in A_i$).

The properties of $X$ and $Y$ listed in the previous paragraph are sufficient to allow us to apply the strong separating hyperplane theorem provided that $X$ and $Y$ are disjoint. We prove this indirectly. Suppose $z \in X \cap Y$. Observe that $z \in X$ implies:

\[(25) \quad y \geq u_i(a^*_i, \langle a_{-i} \rangle).\]

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4An example and a graph that illustrate the modification of Step 2 follow after the end of the proof.
Because $z \in Y$ there is therefore some function $\lambda : A_i \to \mathbb{R}_+$ such that:

\[(26) \quad u_i(a_i^*, \langle a_{-i} \rangle) + \sum_{a_i \in A_i} \lambda(a_i) (u_i(a_i, \langle a_{-i} \rangle) - u_i(a_i^*, \langle a_{-i} \rangle)) \geq u_i(a_i^*, \langle a_{-i} \rangle)\]

which is equivalent to:

\[(27) \quad \sum_{a_i \in A_i} \lambda(a_i) u_i(a_i, \langle a_{-i} \rangle) \geq \sum_{a_i \in A_i} \lambda(a_i) u_i(a_i^*, \langle a_{-i} \rangle)).\]

Because the two sides of (27) are not equal, we have to have $\lambda(a_i) \neq 0$ for at least some $a_i \in A_i$. We can therefore divide both sides of (27) by $\sum_{a_i \in A_i} \lambda(a_i)$ and obtain:

\[(28) \quad \sum_{a_i \in A_i} \frac{\lambda(a_i)}{\sum_{a_i' \in A_i} \lambda(a_i')} u_i(a_i, \langle a_{-i} \rangle) \geq u_i(a_i^*, \langle a_{-i} \rangle))\]

Inequality (28) says that $a_i^*$ is weakly dominated by the mixed action which gives each action in $A_i$ the probability $\lambda(a_i)/\sum_{a_i' \in A_i} \lambda(a_i')$. This contradicts our assumption that $a_i^*$ is not weakly dominated.

We can now conclude using the strong separating hyperplane theorem (Theorem 6 in Border [2] that $X$ and $Y$ can be strongly separated, that is, there are some $\pi \in \mathbb{R}^{|A_i|}$ which is not equal to zero in every component, and some $\delta \in \mathbb{R}$, such that:

\[(29) \quad \pi \cdot x > \delta \quad \forall x \in X\]

and

\[(30) \quad \pi \cdot y \leq \delta \quad \forall y \in Y.\]

We have two observations about $\pi$ and $\delta$. The first is:

\[(31) \quad \delta \geq \pi \cdot u_i(a_i^*, \langle a_{-i} \rangle)).\]

This follows from (30) and $u_i(a_i^*, \langle a_{-i} \rangle) \in Y$. The second observation is:

\[(32) \quad \pi \gg 0.\]

where "$\gg$" means "strictly greater in every component." Suppose (32) were not true, that is, one component of $\pi$, say the $k$-th component, were less than or equal to zero. Denote by $\iota_k$ the unit vector in $\Delta$ that equals 1 in the $k$-th component and zero in all other components. Thus $u_i(a_i^*, \langle a_{-i} \rangle) + \iota_k \in X$. Moreover, by our assumption about $\pi$, we would have:

\[(33) \quad \pi \cdot (u_i(a_i^*, \langle a_{-i} \rangle) + \iota_k) \leq \pi \cdot u_i(a_i^*, \langle a_{-i} \rangle)) \leq \delta\]

where the last inequality is (31). But (33) contradicts (29).

Inequality (32) allows us to normalize $\pi$ so that it becomes a probability vector. We denote this probability vector by $\mu_i$.

\[(34) \quad \mu_i = \frac{\pi}{\|\pi\|}\]
Note that $\mu_i \gg 0$, and therefore we can complete the proof by showing that $a_i^*$ maximizes expected utility in $A_i$ when beliefs are $\mu_i$.

Obviously, inequality (30) holds if we replace $\pi$ by $\mu_i$ and divide the right hand side by $\|\pi\|$.

Now suppose that $a_i^*$ did not maximize expected utility in $A_i$:

\begin{equation}
\mu_i \cdot \left( u_i(a_i, \langle a_{-i} \rangle) - u_i(a_{-i}^*, \langle a_{-i} \rangle) \right) > 0
\end{equation}

for some $a_i \in A_i$. We can find some large enough $\hat{\lambda} > 0$ such that:

\begin{equation}
\mu_i \cdot \hat{\lambda} \left( u_i(a_i, \langle a_{-i} \rangle) - u_i(a_{-i}^*, \langle a_{-i} \rangle) \right) > \mu_i \cdot u_i(a_{-i}^*, \langle a_{-i} \rangle) + \frac{\delta}{\|\pi\|}
\end{equation}

This is equivalent to:

\begin{equation}
\mu_i \cdot \left( \hat{\lambda} \left( u_i(a_i, \langle a_{-i} \rangle) - u_i(a_{-i}^*, \langle a_{-i} \rangle) \right) - u_i(a_{-i}^*, \langle a_{-i} \rangle) \right) > \frac{\delta}{\|\pi\|}
\end{equation}

Now note that $\left( \hat{\lambda} \left( u_i(a_i, \langle a_{-i} \rangle) - u_i(a_{-i}^*, \langle a_{-i} \rangle) \right) - u_i(a_{-i}^*, \langle a_{-i} \rangle) \right) \in Y$ (set $\lambda(a_i) = \hat{\lambda}$ and $\lambda(a'_i) = 0$ for all $a'_i \neq a_i$). Thus, we have found an element $y$ of $Y$ such that $\mu_i \cdot y > \delta/\|\pi\|$, which contradicts (35).

We illustrate Step 2 of the proof of Proposition 2 with the same game that we used above to illustrate the proof of Proposition 1. In that game, player 1’s action $A$ is not only not strictly dominated, but also not weakly dominated. We illustrate in Figure 3 the construction of the sets $X$ and $Y$ to which the proof of Proposition 2 refers.
The set $Y$ is now the cone generated by the set that we had called $Y$ in Figure 1. In Figure 3 this cone is indicated by parallel lines from the top left to the bottom right. The set $X$ is the unbroken line in the bottom left corner of the shaded rectangle that represented $X$ in Figure 1. The hyperplane separating $X$ and $Y$ is the dashed line in Figure 3. Figure 3 also shows the orthogonal vector for this hyperplane.

4. Iterated Elimination of Strictly Dominated Actions

If players know each others’ utility functions, and know that other players are rational, and know that other players know that players are rational, etc., it seems plausible to iterate the elimination of strictly dominated actions.

**Definition 6.** A finite sequence $((X^t_i)_{i \in N})_{t=0}^{T}$ (where $T \in \mathbb{N}$) is a process of iterated elimination of strictly dominated actions (IESDA) if for all $i \in N$:

(i) $X^0_i = A_i$

c and for all $i \in N$ and $t = 0, 1, \ldots, T - 1$:

(ii) $X^{t+1}_i \subseteq X^t_i$

(iii) $a_i \in X^t_i \setminus X^{t+1}_i$ only if $a_i$ is strictly dominated in the strategic game with player set $N$, action sets $X^t_j$ for every $j \in N$, and utility functions equal the restrictions of $u^k$ to $\times_{j \in N} X^t_j$.

(iv) if $a_i \in X^T_i$ then $a_i$ is not strictly dominated in the strategic game with player set $N$, action sets $X^T_j$ for every $j \in N$, and utility functions equal to the restrictions of $u^k$ to $\times_{j \in N} X^T_j$.

Note that there are multiple processes of IESDA. However, as we show below, they all terminate with the same sets of actions, and therefore the phrase “actions that survive IESDA” is unambiguously defined. We first need a preliminary lemma.

**Lemma 1.** Let $((X^t_j)_{j \in N})_{t=0}^{T}$ be a process of IESDA. Suppose $a_i \in X^T_i$ is a best reply to some belief $\mu_i \in \Delta(X^T_i)$ among all actions in $X^T_i$. Then $a_i$ is also a best response to $\mu_i$ among all actions in $A_i$.

**Proof.** Let $\tilde{a}_i$ be any best response to $\mu_i \in \Delta(X^T_i)$ among all actions in $A_i$. Then $\tilde{a}_i$ can not be eliminated by IESDA in any step $t$, because of the equivalence of not being strictly dominated and being a best response to some belief, because $\mu_i$ having support in $\Delta(X^T_i)$ implies that $\mu_i$ has support in $\Delta(X^T_i)$, and because $\tilde{a}_i$ is a best response to $\mu_i$ among all actions in $X^T_i$. Therefore, $\tilde{a}_i \in X^T_i$. If $a_i$ is a best response to $\mu_i$ in $X^T_i$, it must therefore yield at least as high expected utility, and hence equilibrium the

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5I.e. utility functions and rationality are common knowledge.
same expected utility, as $\tilde{a}_i$. Therefore, it is also a best response to $\mu_i$ among all actions in $A_i$. \hfill \Box

**Proposition 3.** Let $((X^t_i)_{i \in N})_{t=0}^T$ and $((\hat{X}^t_i)_{i \in N})_{t=0}^\hat{T}$ both be processes of IESDA. Then $X^T_i = \hat{X}^\hat{T}_i$ for all $i \in N$.

**Proof.** It suffices to prove:

$$\hat{X}^T_i \subseteq X^T_i \forall t = 0, 1, \ldots, T.$$  

This is sufficient because it implies $\hat{X}^T_i \subseteq X^T_i$, and, by the symmetric argument, we can also infer $X^T_i \subseteq \hat{X}^T_i$, so that the assertion follows. We prove (39) by induction over $t$. The assertion is trivially true for $t = 0$. For the induction step we have to prove that $\hat{X}^T_i \subseteq X^t_i$ for all $i \in N$ implies $\hat{X}^T_i \subseteq X^{t+1}_i$ for all $i \in N$. Consider some $i \in N$ and some $a_i \in \hat{X}^T_i$. We have to show $a_i \in X^{t+1}_i$. Because $a_i$ is not strictly dominated in the game left over at the end of the process $((\hat{X}^t_i)_{i \in N})_{t=0}^\hat{T}$, it is a best response among all actions in $A_i$ to a belief $\mu_i \in \Delta(\hat{X}^\hat{T} - i)$ (using the above Lemma). By the inductive assumption, $\mu_i$’s support is included in $X^{t-1}_i$. Because $a_i$ is a best response to $\mu_i$ among all actions in $A_i$, it is also a best response to $\mu_i$ among the actions in in $X^t_i$. Therefore, it cannot be eliminated in step $t$ of the process $((X^t_i)_{i \in N})_{t=0}^T$, and therefore $a_i \in X^{t+1}_i$. \hfill \Box

Adapting Definition 6 appropriately, one might also define the iterated elimination of weakly dominated actions. The analog of Proposition 3 is, however, not correct when iterated elimination of weakly dominated actions is considered. For this and other reasons iterated elimination of weakly dominated actions is a problematic solution concept that we do not consider any further in these notes.

5. Other Elimination Processes: Iterated Elimination of Weakly Dominated Actions; Rationalizability

Instead of iterating the elimination of strictly dominated actions, we might feel tempted to iterate the elimination of weakly dominated actions. Iterated elimination of weakly dominated strategies is, however, a problematic procedure. A first sign that there is a problem with this procedure is the fact that the order in which we eliminate weakly dominated strategies might matter for the result, i.e. Proposition j3 does not apply to iterated elimination of weakly dominated strategies. The simplest example that illustrates this is as follows.
In this example, if we first eliminate player 2’s weakly dominated action $R$, but none of player 1’s actions, then we cannot eliminate any further actions in subsequent rounds, and hence both $T$ and $B$ are left over for player 1. But if we first eliminate player 1’s weakly dominated action $B$, and none of player 2’s actions, then we can again not eliminate any further actions in subsequent rounds, and both $L$ and $R$ are left over for player 2. Thus, our conclusion depends on the order of elimination.

The deeper reason for this problem is that unlike IESDA, the iterated elimination of weakly dominated actions is not based on a common knowledge assumption. Weakly dominated strategies can only be eliminated if a player attaches positive probability to all strategy combinations of the other players, but then the player can not know any statement that has non-trivial implications for the other players’ action choice.

There is one other iterated elimination procedure that we should mention. Suppose we assume that every player believes that the other players’ actions are “independent.” Formally, suppose we require that any player $i$’s belief about two other players $j$’s and $k$’s actions is the product of the two marginal measures on player $j$’s and player $k$’s action sets, then Proposition 1 is no longer true in all games. There are games in which more strategies can be eliminated if we require that beliefs are the products of their marginals than if we allow beliefs to reflect arbitrary correlations. Obviously, in such games there must be at least 3 players. There is no simple characterization in terms of some “dominance” notion of the property of being a best response to a product belief. However, the procedure of iterated elimination of actions that are not best responses to product beliefs is well-defined. Moreover, it can be shown (easily) that the outcome of this procedure is order-independent, unlike the outcome of iterated deletion of weakly dominated strategies. The actions that are left at the end of this procedure are called “rationalizable.” We omit here a further discussion of rationalizability.

6. Nash Equilibrium

We now come to the most classic solution concept for strategic games.
Definition 7. A list of mixed actions \( \alpha \in \times_{i \in N} \Delta(A_i) \) is a Nash equilibrium if for every player \( i \in N \):

\[
\alpha_i \in \arg \max_{\alpha_i' \in \Delta(a_i)} U_i(\alpha_i', \alpha_{-i})
\]

It is useful to re-write the definition of Nash equilibrium in the following way. For every player \( i \) and for every \( \alpha_{-i} \in \times_{j \in N, j \neq i} \Delta(a_j) \) define the set of best replies of player \( i \):

\[
B_i(\alpha_{-i}) = \arg \max_{\alpha_i' \in \Delta(A_i)} U_i(\alpha_i', \alpha_{-i}).
\]

For every \( \alpha \in \times_{j \in N} \Delta(A_j) \) define:

\[
B(\alpha) = \times_{i \in N} B_i(\alpha_{-i}).
\]

A Nash equilibrium is then a fixed point of the correspondence \( B \), that is, an \( \alpha \) such that \( \alpha \in B(\alpha) \).

The following result gives a characterization of best responses that is useful when determining Nash equilibria.

Lemma 2. For every player \( i \) and every \( \alpha_{-i} \) we have \( \alpha_i \in B_i(\alpha_{-i}) \) if and only if \( \alpha_i(a_i) > 0 \) implies \( a_i \in \arg \max_{a_i' \in A_i} U_i(a_i', \alpha_{-i}) \).

Proof. The same argument that leads to (5) in Step 1 of the proof of Proposition 1 shows that for every \( \alpha_i \in \Delta(A_i) \):

\[
U_i(\alpha_i, \alpha_{-i}) = \sum_{a_i \in A_i} \alpha_i(a_i) \left( \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \prod_{j \in N, j \neq i} \alpha_j(a_j) \right).
\]

This implies:

\[
U_i(\alpha) \leq \max_{a_i \in A_i, \alpha_i(a_i) > 0} U_i(a_i, \alpha_{-i})
\]

which in turn implies:

\[
U_i(\alpha) \leq \max_{a_i \in A_i} U_i(a_i, \alpha_{-i}).
\]

Therefore:

\[
\max_{\alpha_i \in \Delta(A_i)} U_i(\alpha_i, \alpha_{-i}) \leq \max_{a_i \in A_i} U_i(a_i, \alpha_{-i}).
\]

On the other hand, because \( a_i \in \Delta(A_i) \) for all \( a_i \in A_i \):

\[
\max_{\alpha_i \in \Delta(a_i)} U_i(\alpha_i, \alpha_{-i}) \geq \max_{a_i \in A_i} U_i(a_i, \alpha_{-i}).
\]

Therefore,

\[
\max_{\alpha_i \in \Delta(a_i)} U_i(a_i, \alpha_{-i}) = \max_{a_i \in A_i} U_i(a_i, \alpha_{-i}).
\]
which means that $\alpha_i$ is a best response to $\alpha_{-i}$ if and only if

$$U_i(\alpha) = \max_{a_i \in A_i} U_i(a_i, \alpha_{-i}).$$

The proof is completed by observing that (43) shows that whenever $a_i \in \arg \max_{a_i' \in A_i} U_i(a_i', \alpha_{-i})$ for all $\alpha_i(a_i) > 0$ equation (49) holds. By contrast, if there is some $a_i \notin \arg \max_{a_i' \in A_i} U_i(a_i', \alpha_{-i})$ with $\alpha_i(a_i) > 0$ then $U_i(\alpha_i, \alpha_{-i}) < \max_{a_i \in A_i; \alpha_i(a_i) > 0} U_i(a_i, \alpha_{-i})$. □

We now prove a preliminary result that then allows us to prove the existence theorem for Nash equilibria.

**Lemma 3.** For every player $i$ and every $\alpha_{-i}$ the set $B_i(\alpha_{-i})$ is non-empty and convex.

**Proof.** $B_i(\alpha_{-i})$ is non-empty because among the finitely many elements of $A_i$ at least one must be a best response among all elements of $A_i$ to $\alpha_{-i}$ in $A_i$, and by Lemma 2 this action is contained in $B_i(\alpha_{-i})$. To see that $B_i(\alpha_{-i})$ is convex, assume $\alpha_i, \alpha'_i \in B_i(\alpha_{-i})$ and $0 < \lambda < 1$. We have to show: $\alpha_i^\lambda = \lambda \alpha_i + (1 - \lambda)\alpha'_i \in B_i(\alpha_{-i})$. By Lemma 2 this means that we have to show $\alpha_i^\lambda(a_i) > 0$ implies that $a_i \in \arg \max_{a_i' \in A_i} U_i(a_i', \alpha_{-i})$. But $\alpha_i^\lambda(a_i) > 0$ implies that either $\alpha_i(a_i) > 0$, or $\alpha'_i(a_i) > 0$, or both. Because $\alpha_i$ and $\alpha_{-i}$ are best responses to $\alpha_{-i}$, we can apply Lemma 2 to at least one of these mixed actions, and obtain directly what we have to show. □

**Proposition 4.** Every strategic game has at least one Nash equilibrium.

**Proof.** It suffices to show that $B$ satisfies the conditions of Kakutani’s fixed point theorem (Lemma 20.1 in Osborne and Rubinstein [1]), which guarantees the existence of at least one fixed point. For this we have to verify that the domain of $B$ is compact and convex, that $B(\alpha)$ is non-empty and convex for every $\alpha$, and that $B$ has a closed graph. That the conditions for the domain are satisfied is obvious. Non-emptiness and convexity of $B(\alpha)$ for every $\alpha$ follow from Lemma 3. That $B$ has a closed graph follows from the maximum theorem (Theorem 9.14 in Sundaram [4]). □

Finally, we relate Nash equilibrium actions to actions that survive IESDA.

**Proposition 5.** Suppose that $\alpha$ is a Nash equilibrium, and that for some $i \in N$ and $a_i \in A_i$ we have: $\alpha_i(a_i) > 0$. Then $a_i$ survives IESDA.

**Proof.** Let $((X^t_i)_{i \in N})_{t=0}^T$ be a process of IESDA. We shall show that $a_i \in X^t_i$ for all $t = 0, 1, \ldots, T$. We prove this by induction over $t$. The assertion is obvious for $t = 0$. Now suppose we had shown the assertion for $t$. We want
to prove it for \( t + 1 \). By Proposition 1 it is sufficient to show that \( a_i \) is a best response to some belief \( \mu_i \) over \( \times_{j \in N, j \neq i} X_j^t \). Define this belief by:

\[
\mu_i(a_{-i}) = \prod_{j \in N, j \neq i} \alpha_j(a_j).
\]

This is a probability measure over \( \times_{j \in N, j \neq i} X_j^t \) because, by the inductive assumption, \( \alpha_j(a_j) > 0 \) implies \( a_j \in X_j^t \). Moreover, \( a_i \) is a best response because by assumption \( \alpha \) is a Nash equilibrium, and therefore, by Lemma 2 every pure action that is contained in the support of \( \alpha_i \) is a best response to \( \alpha_{-i} \).

One can also easily show that every action that is played with positive probability in a Nash equilibrium is rationalizable.

### 7. Correlated Equilibrium

The following equilibrium concept, “correlated equilibrium” is intermediate between Nash equilibrium and IESDA. It is less commonly used in applications. Yet, when one considers the epistemic or evolutionary foundations of game theoretic solution concepts, we find that there seem strong arguments in favor of correlated equilibrium.

We begin with some notation. Let \( \gamma \) be some probability distribution over \( A \). We shall denote by \( \gamma_i \in \Delta(A_i) \) the marginal distribution of \( a_i \), that is, for every \( a_i \in A_i \):

\[
\gamma_i(a_i) = \sum_{a_{-i} \in A_{-i}} \gamma(a_i, a_{-i}).
\]

Consider any action \( a_i \in A_i \) with \( \gamma_i(a_i) > 0 \). We shall denote by \( \gamma(a_{-i}|a_i) \) the conditional distribution of \( a_{-i} \), conditional on \( a_i \). That is, for every \( a_{-i} \in A_{-i} \) we have:

\[
\gamma_i(a_{-i}|a_i) = \frac{\gamma(a_i, a_{-i})}{\gamma_i(a_i)}.
\]

Finally, for every arbitrary action \( a'_i \in A_i \) we shall denote by \( U_i(a'_i|a_i) \) the expected utility of player \( i \) if player \( i \) chooses action \( a_i \) and the other players’ actions are distributed according to \( \gamma(\cdot|a_i) \), that is:

\[
U_i(a'_i|a_i) = \sum_{a_{-i} \in A_{-i}} u_i(a'_i, a_{-i}) \gamma(a_{-i}|a_i).
\]

Equipped with this notation, we can now state the following definition.

**Definition 8.** A correlated equilibrium is a probability distribution \( \gamma \) over \( A \) such that for every player \( i \), for every action \( a_i \in A_i \) such that \( \gamma_i(a_i) > 0 \) we have:

\[
U_i(a_i|a_i) \geq U_i(a'_i|a_i) \quad \text{for all } a'_i \in A_i.
\]
To understand this definition intuitively, imagine a mediator who recommends to players which action to choose. The mediator proceeds as follows. First, she chooses an action profile $a \in A$ according to some probability distribution $\gamma$. She does not reveal $a$ completely to all players. Rather, she only reveals for each player $i$ which action $a_i$ she has drawn for that agent $i$. Think of this action $a_i$ as the recommendation that the mediator makes to player $i$. If player $i$ knows the distribution $\gamma$, and trusts that the mediator is truthful, then the inequality in Definition 8 says that agent $i$, when receiving the recommendation to play $a_i$, maximizes her expected utility by following the mediator’s recommendation. Here, when calculating her expected utility, she infers whatever she can about the distribution of the other player’s actions $a_{-i}$ from the fact that she has been recommended to play $a_i$.

Let us illustrate the concept of correlated equilibrium with the following example:

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>6,6</td>
<td>2,7</td>
</tr>
<tr>
<td>T</td>
<td>7,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Example 3

Let $\gamma$ be the distribution that assign probability 1/3 each to the strategy combinations $(S, S)$, $(S, T)$, and $(T, S)$, but probability 0 to the strategy combination $(T, T)$. Then $\gamma$ is a correlated equilibrium. Consider player 1. When he receives the recommendation to play $S$, the distribution $\gamma(\cdot|S)$ assigns probability 1/2 each to the actions $S$ and $T$ of player 2. Given this distribution, player 1’s expected utility from playing $S$ is $(6 + 2)/2 = 4$, and her expected utility from playing $T$ is: $(7 + 0)/2 = 3.5$. Thus, she finds it in her interest to follow the recommendation. When player 1 receives the recommendation to play $T$, the distribution $\gamma(\cdot|T)$ assigns probability 1 to the action $S$ of player 2. Therefore, player 1’s expected utility from playing $S$ is 6, but her expected utility from playing $T$ is 7. Therefore, again, she finds that her expected utility is maximized if she follows the recommendation.

The game in Example 3 is known as the game of “Chicken,” where $S$ stands for acting “softly,” and $T$ stands for acting “tough.” The game has three Nash equilibria: $(S, T)$, $(T, S)$, and the mixed strategy equilibrium in which every player chooses $S$ with probability 2/3 and $T$ with probability 1/3. In the mixed strategy Nash equilibrium the undesirable outcome that both players choose $T$ occurs with probability 1/9. By contrast, the correlated equilibrium completely avoids this outcome.

We first note that every Nash equilibrium in pure or mixed strategies is a correlated equilibrium in the following sense:
Proposition 6. If $\alpha$ is a Nash equilibrium, then the distribution $\gamma$ given by:
\begin{equation}
\gamma(a) = \alpha_1(a_1) \cdot \alpha_2(a_1) \cdot \ldots \cdot \alpha_n(a_n) \text{ for all } a \in A
\end{equation}
is a correlated equilibrium.

Proof. When $\gamma$ is defined as described in Proposition 6, then obviously $\gamma_i = \alpha_i$ for every player $i$, and therefore $U_i(\cdot|a_i) = U_i(\cdot, \alpha_{-i})$ for every $a_i \in A_i$. But it then follows directly from Lemma 2 that every player $i$ maximizes conditional expected utility when recommended action $a_i$, as long as $\gamma_i(a_i) > 0$. \hfill \square

The next result describes easily derived properties of the set of all correlated equilibria.

Proposition 7. For every strategic game the set of correlated equilibria is a non-empty and convex set.

Proof. The set of correlated equilibria is non-empty because, by Proposition 6, every Nash equilibrium is a correlated equilibrium, and by Proposition 4, every strategic game has a correlated equilibrium. It remains to show that the set of correlated equilibria is convex, that is: if $\gamma$ and $\gamma'$ are correlated equilibria, and if $\lambda \in (0, 1)$, then also $\lambda \gamma + (1 - \lambda) \gamma'$ is a correlated equilibrium. What we have to show is that for every player $i$, and for every $a_i \in A_i$ for which $\lambda \gamma_i(a_i) + (1 - \lambda) \gamma'_i(a_i) > 0$ we have:
\begin{equation}
\sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \frac{\lambda \gamma(a_i, a_{-i}) + (1 - \lambda) \gamma'(a_i, a_{-i})}{\lambda \gamma_i(a_i) + (1 - \lambda) \gamma'_i(a_i)} \geq \sum_{a_{-i} \in A_{-i}} u_i(a_i', a_{-i}) \frac{\lambda \gamma(a_i', a_{-i}) + (1 - \lambda) \gamma'(a_i', a_{-i})}{\lambda \gamma_i(a_i') + (1 - \lambda) \gamma'_i(a_i')}
\end{equation}
for all $a'_i \in A_i$. Multiplying both sides by $\lambda \gamma_i(a_i) + (1 - \lambda) \gamma'_i(a_i)$, we can re-write what we have to show as:
\begin{equation}
\sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \left( \lambda \gamma(a_i, a_{-i}) + (1 - \lambda) \gamma'(a_i, a_{-i}) \right) \geq \sum_{a_{-i} \in A_{-i}} u_i(a_i', a_{-i}) \left( \lambda \gamma(a_i', a_{-i}) + (1 - \lambda) \gamma'(a_i', a_{-i}) \right).
\end{equation}
This can be re-written as:
\begin{equation}
\lambda \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \gamma(a_i, a_{-i}) + (1 - \lambda) \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \gamma'(a_i, a_{-i}) \geq \lambda \sum_{a_{-i} \in A_{-i}} u_i(a'_i, a_{-i}) \gamma(a_i, a_{-i}) + (1 - \lambda) \sum_{a_{-i} \in A_{-i}} u_i(a'_i, a_{-i}) \gamma'(a_i, a_{-i}).
\end{equation}
This follows from:

$$\sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \gamma(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} u_i(a_i', a_{-i}) \gamma(a_i, a_{-i})$$

and

$$\sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \gamma'(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} u_i(a_i', a_{-i}) \gamma'(a_i, a_{-i}).$$

(59)

The first inequality is trivially true if $\gamma_i(a_i) = 0$, because then both sides of the inequality equal zero. But if $\gamma_i(a_i) > 0$, then we can divide the first inequality by $\gamma_i(a_i)$, and obtain:

$$\sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \gamma(a_i, a_{-i}) \gamma_i(a_i) \geq \sum_{a_{-i} \in A_{-i}} u_i(a_i', a_{-i}) \gamma(a_i, a_{-i}) \gamma_i(a_i),$$

which is true because $\gamma$ is a correlated equilibrium. The same argument proves the second inequality.

We conclude this section by proving that Proposition 5 extends to correlated equilibria.

**Proposition 8.** Suppose that $\gamma$ is a correlated equilibrium, and that for some $i \in N$ and $a_i \in A_i$ we have: $\gamma_i(a_i) > 0$. Then $a_i$ survives IESDA.

**Proof.** Let $((X^t_j)_{j \in N})_{t=0}^T$ be a process of IESDA. We shall show that $a_i \in X^t_i$ for all $t = 0, 1, \ldots, T$. We prove this by induction over $t$. The assertion is obvious for $t = 0$. Now suppose we had shown the assertion for $t$. We want to prove it for $t + 1$. By Proposition 1 it is sufficient to show that $a_i$ is a best response to some belief $\mu_i$ over $\times_{j \in N, j \neq i} X^t_j$. Define this belief by: $\mu_i = \gamma(\cdot | a_i)$. This is a probability measure over $\times_{j \in N, j \neq i} X^t_j$ because $\gamma(a_j | a_i) > 0$ implies $\gamma_j(a_j) > 0$, and therefore by the inductive assumption $a_j \in X^t_j$. Moreover, $a_i$ is a best response to $\mu_i$ by the definition of correlated equilibrium.

Propositions 6 and 8 show that the concept of correlated equilibrium is intermediate between Nash equilibrium and IESDA.

We now calculate as an illustration for the game in Example 3 the set of all correlated equilibria. We shall denote probability distributions over $A$ in that example by $(p, q, r, s) \in \mathbb{R}_+$ where $p + q + r + s = 1$, and where the interpretation of the probabilities is indicated in the following table:

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>T</th>
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</thead>
<tbody>
<tr>
<td>S</td>
<td>p</td>
<td>q</td>
</tr>
<tr>
<td>T</td>
<td>r</td>
<td>s</td>
</tr>
</tbody>
</table>
When is a vector \((p, q, r, s)\) a correlated equilibrium? Consider the condition that ensures that player 1, when the mediator recommends that he play \(S\), finds it indeed in his interest to play \(S\). Conditional on the recommendation \(S\) is expected utility maximizing if:

\[
6p + 2q \geq 7p
\]

which is equivalent to \(p \leq 2q\). This is the first of four conditions that are necessary and equivalent for \((p, q, r, s)\) to be a correlated equilibrium. We list all conditions below, and leave it to the reader to derive the other three conditions.

\[
\begin{align*}
(61) & \quad p \leq 2q \\
(62) & \quad p \leq 2r \\
(63) & \quad s \leq 0.5q \\
(64) & \quad s \leq 0.5r
\end{align*}
\]

The set of correlated equilibria is convex, as we saw above, and therefore, it is of interest to determine its extreme points. In fact, by the Krein-Milman theorem, the set of correlated equilibria is the convex hull of its extreme points. The following result describes the extreme points.

**Claim:** The set of correlated equilibria of the game of Chicken has five extreme points:

\[
(0, 1, 0, 0), (0, 0, 1, 0), \left(\frac{4}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}\right), \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0\right), \text{ and } \left(0, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right).
\]

Observe that the first two extreme points are the two pure strategy Nash equilibrium, and that the third extreme point is the mixed strategy Nash equilibrium of Chicken. Because the set of correlated equilibria is the convex hull of the five correlated equilibria listed in the claim, one can implement any arbitrary correlated equilibrium by writing it as a convex combination of the extreme points, picking one of the “extreme” correlated equilibria at random, with the probabilities of each equilibrium determined by its weight in the convex combination, then announcing which of the extreme equilibria has been picked, and then giving each player an action recommendation where the recommendations going to both players are jointly determined by a random draw according to the chosen “extreme” correlated equilibrium.

**Proof:** We first show that no other points are extreme points of the set of correlated equilibrium, and then we show that these five points are all extreme points.

Suppose first \(q + r = 1\). All such probability vectors are correlated equilibria. We argue that in an extreme point either \(q\) or \(r\) must be zero. This
is because, if both $q$ and $r$ are positive, then one can find an $\epsilon > 0$ such that $(0, q + \epsilon, r - \epsilon, 0)$ and $(0, q - \epsilon, r + \epsilon, 0)$ are also correlated equilibria, and because the given correlated equilibrium is the convex combination with equal weights of the two new correlated equilibrium, we don’t have an extreme point.

Consider next $q + r < 1$. We begin by noting that in any correlated equilibrium we then must have that both $q > 0$ and $r > 0$. To see this suppose w.l.o.g. $q = 0$. Then it cannot be that player 2 is recommended to play $T$ with positive probability, because she would then know that player 1 has been recommended $T$ as well, and she would have an incentive to deviate. So player 2 plays $S$ with probability 1. But then it must be that player 1 plays $T$ with probability 1. She would otherwise always deviate to $T$. Therefore, it must be that the Nash equilibrium $(T, S)$ is played with probability 1, which contradicts $q + r < 1$. We conclude that we must have $q > 0$ and $r > 0$.

Now suppose $q + r < 1$, $q > 0$, $r > 0$, $p > 0$ and $s > 0$. This can be an extreme point only if $q = r$. To see this, suppose w.l.o.g. $q < r$. Then, for sufficiently small $\epsilon > 0$, both $(p - 2\epsilon, q - \epsilon, r + 3.5\epsilon, s - 0.5\epsilon)$ and $(p + 2\epsilon, q + \epsilon, r - 3.5\epsilon, s + 0.5\epsilon)$ are correlated equilibria, and the original correlated equilibrium is the convex combination with equal weights of these two new correlated equilibria, and therefore cannot be an extreme point. We also note that we must have $p = 2q$, because, otherwise, for sufficiently small $\epsilon$, both $(p - 1.5\epsilon, q + \epsilon, r + \epsilon, s - 0.5\epsilon)$ and $(p + 1.5\epsilon, q - \epsilon, r - \epsilon, s + 0.5\epsilon)$ would be correlated equilibria, and we could not have an extreme point. Similarly, we must have $s = 0.5q$. From $q = r$, $p = 2q$ and $s = 0.5q$, we conclude that the only candidate for an extreme point of the set of correlated equilibria that satisfies $q + r < 1$, $q > 0$, $r > 0$, $p > 0$ and $s > 0$ is: $(\frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5})$.

Suppose next $q + r < 1$, $q > 0$, $r > 0$, $p = 1 - q - r$, and $s = 0$. Using arguments as in the previous paragraph one shows that we must have $q = r$, and also $p = 2q$. But the system $q = r$, $p = 2q$, and $q + r + p = 1$ has just one solution: $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$.

Now suppose $q + r < 1$, $q > 0$, $r > 0$, $p = 0$, and $s = 1 - q - r$. By similar arguments as in the previous paragraph we then have an extreme point only if $q = r$ and $s = 0.5q$. This leads to the solution: $(0, \frac{2}{5}, \frac{2}{5}, \frac{1}{5})$.

We conclude that five points listed in the claim are the only candidates for extreme points of the set of correlated equilibria. It remains to show that all of them are indeed extreme points. But if one of them were not an extreme point, then by the Krein-Milman theorem, it would have to be possible to write one of them as a convex combination of the other points. That this is impossible follows from straightforward arguments that we omit.

Q.E.D.
We can now determine the set of payoff vectors that correspond to correlated equilibria. The five payoff vectors that correspond to the five extreme points of the set of correlated equilibria are:

$$(2, 7), (7, 2), \left(\frac{14}{3}, \frac{14}{3}\right), \left(\frac{21}{4}, \frac{21}{4}\right), \text{ and } \left(\frac{18}{5}, \frac{18}{5}\right).$$

Note that these are the only candidates for extreme points of the set of payoff vectors that correspond to correlated equilibria, but that not all of these need to be extreme points. Indeed, the mixed strategy Nash equilibrium payoff vector is clearly a convex combination of the last two payoff vectors, and therefore not an extreme point. The remaining four are all extreme points of the set of correlated equilibrium payoff vectors. The set is shown in the following figure. All payoff vectors corresponding to extreme points of the set of correlated equilibria are marked by small circles in this figure.

![Figure 4](image)

We ask next which of the correlated equilibria are on the boundary of the set of correlated equilibria in this example. We say that a vector $(p, q, r, s)$ is on the boundary of the set of correlated equilibria if every open ball around $(p, q, r, s)$ contains a probability vector that is not a correlated equilibrium. Recall that every correlated equilibrium can be represented as a convex combination of the five extreme correlated equilibria listed above. Thus, we can represent correlated equilibria by indicating their weights $(w_1, w_2, \ldots, w_5)$ with $w_k \geq 0$ for all $k$ and $w_1 + w_2 + \ldots + w_5 = 1$, where $w_1$ is the weight on $(0, 1, 0, 0)$, $w_2$ is the weight on $(0, 0, 1, 0)$, etc. We shall use this representation in the statement of our result about the boundary points of the set of correlated equilibria.
Claim: \((w_1, w_2, \ldots, w_5)\) is on the boundary of the set of correlated equilibria if and only if both of the following two conditions hold:

(i) \(w_1w_2 = 0\) (i.e. at most one of \(w_1\) and \(w_2\) is non-zero);

(ii) \(w_4w_5 = 0\) (i.e. at most one of \(w_4\) and \(w_5\) is non-zero).

Proof: One can prove first that a correlated equilibrium is on the boundary if and only if there is an action that is recommended with probability zero, or that is recommended with positive probability and for which the incentive constraint binds. One can prove then, second, that there is an action that is recommended with probability zero, or that is recommended with positive probability and for which the incentive constraint binds if and only if (i) and (ii) of the Claim hold. (Details to be filled in later.)

Q.E.D.

References


