

When Are Signals Complements or Substitutes?*

Tilman Börgers[†], Angel Hernando-Veciana[‡], and Daniel Krähmer[§]

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[†]Department of Economics, University of Michigan, tborgers@umich.edu.

[‡]Departament of Economics Universidad Carlos III de Madrid, angel.hernando@uc3m.es

[§]Institut für Wirtschaftstheorie, Freie Universität Berlin, Daniel.Kraehmer@wiwiss.fu-berlin.de.

Abstract

The paper introduces a notion of complementarity (substitutability) of two signals which requires that in all decision problems each signal becomes more (less) valuable when the other signal becomes available. We provide a general characterization which relates complementarity and substitutability to a Blackwell-comparison of two auxiliary signals. In a special setting with a binary state space and symmetric binary signals, we find an explicit characterization that permits an intuitive interpretation of complementarity and substitutability. We demonstrate how these conditions extend to the general case. Finally, we study implications of complementarity and substitutability for information acquisition, for information revelation by a monopolist, and in a second price auction.

Keywords: Complementarity, substitutability, value of information, Blackwell-ordering, statistical decision problem, information acquisition, second price auction

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1 Introduction

Suppose that two signals are available to a decision maker, and that each signal contains some information about an aspect of the world that is relevant to a future decision. In this paper we ask under which conditions these two signals are *substitutes*, and under which conditions they are *complements*. Roughly speaking, we mean by this that the incentive to acquire one signal decreases as the other signal becomes available (in the case of substitutes), or that it increases as the other signal becomes available (in the case of complements).

Now the incentives to acquire signals depend, of course, on the decision for which the information will be used. When we call signals complements or substitutes in this paper, then we mean that the conditions described above are satisfied *in all decision problems*. Hence we say in this paper that signals are substitutes if in all decision problems the value of each signal decreases as the other signal becomes available. The signals are complements if in all decision problems the value of each signal increases as the other signal becomes available. The conditions that we shall provide will thus not refer to any particular decision problem, but only to the joint distribution of signals, conditional on the various possible states of the world. We identify statistical features of signals which imply that these signals are substitutes or complements.¹

We now give a simple example that indicates how signals can be complements. Suppose that you can observe in a war the enemy's coded communication, and that you have access to the enemy's encryption code. Then observing the enemy's communication is of no use if you do not know how it is coded, and understanding the encryption code is of no use if you don't have any access to communication that uses this code. However, together the two pieces of information are potentially valuable. Your incentive to acquire any one is larger if you already have the other.

One of the main results in this paper shows that, in a particular and special setting, complementary signals are characterized by a property that is very closely related to the main feature of the above example. This property is that the meaning of a realization of one signal depends on the realization

¹Our approach is in the spirit of Blackwell's (1951) comparison of the informativeness of signals. There are also close formal connections which we shall explain in this paper.

of the other signal. The second signal thus provides the key that is needed to unlock the first signal. More technically, the result shows that, in a specific setting, signals are complements if and only if there is a realization of one signal that may increase, but also decrease, the decision maker's subjective probability that some event has occurred, depending on what the other signal is.

The experience that one signal can change the meaning of another signal is familiar. A long wait for a response can be good or bad news for a job applicant. If it is on balance, a priori, good news, then information about a speedily proceeding selection process may turn it into bad news. Seeing a person in the street asking for money to pay for their accidentally lost train ticket may signal on balance that this individual has indeed encountered a temporary difficulty. Seeing the same person making the same request twice, on two different days, may overturn this, and suggest that the story is made up.²

Dow and Gorton (1993) give the example of a technology company that is observed by two analysts. One analyst learns whether the company's lead engineer is leaving the company to create an independent competitor, and the other learns whether the technology that the engineer is working on is likely to succeed. If the technology is likely to succeed and the engineer stays, then this is good news. If the technology is likely to fail, and the engineer leaves, that is also good news because the company is likely to stay dominant in its market. However, the remaining cases are bad news, because either a competitor with a promising technology is created, or because a dubious project will be continued further. The interpretation of each analyst's signal may be reversed by the other analyst's signal.

As our final example recall the common experience that sometimes the news is "too good to be true." One telephone call telling you that you have won the lottery might be interpreted as genuine good news. Two telephone calls, telling you that you have won two different lotteries, strongly suggests that you are the target of some fraud.

The strong reversal result that we have just illustrated will be shown in this paper for a special setting only. However, we also explore the extent to which it generalizes. We show that in many cases it is necessary for

²Note that in the second example signals are symmetric, as they will be in the special case for which we obtain the result illustrated here.

complementarity of signals that the meaning of the realization of one signal can be reverted by a realization of the other signal. We cannot show, however, that this condition is sufficient.

We also explore whether there are cases where pairs of signals are complements without having the property that one signal's meaning can be reverted by the other signal. A case in point is when one signal is completely uninformative about the state of the world and is thus useless by itself. Yet, as we shall demonstrate, such a signal might still enhance the value of the other signal by providing information about the other signal's quality.

A simple example that indicates how signals can be substitutes is also easily constructed. Suppose you have two advisors, and you know that they both work with the same sources, and will tell you exactly the same thing. Then each of them will have positive value, but once you have heard what one of them says, you do not derive any additional benefit from hearing what the other one says.

For the special setting referred to earlier we shall show that signals are substitutes if and only if they share one important feature with the example described in the previous paragraph, namely that the value of a second signal will always be zero. In a more general setting, a related, less stringent condition is a necessary condition for signals to be substitutes. This necessary condition is that the additional signal cannot reinforce the decision maker's most extreme belief that he can have after observing one signal alone.

The results described so far provide interesting, yet partial insights into the nature of the complementarity and substitutability relations among signals. We also offer in this paper completely general characterizations of complements and substitutes. These results show that two signals are complements (resp. substitutes) if and only if, among two other signals that are derived from the two original signals, one dominates the other in the sense of Blackwell (1951), that is, is more valuable in all decision problems. We thus reduce the problem of determining whether two signals are complements (resp. substitutes) to the problem of determining whether among two other signals one Blackwell- dominates the other. This is useful because it allows us to use well-known characterizations of Blackwell-dominance to find out whether two signals are complements (resp. substitutes).

Complementarity and substitutability of signals are also important if different signals are accessible to different decision makers. We will elaborate

below the economic relevance of the relations among signals that we investigate in this paper in three examples of environments with decentralized information. In the first example, we consider a strategic information acquisition game in which each decision maker can observe the other player's information before making a decision. In the second example, we study the incentives of a seller of a single indivisible good to reveal information about the good when this information is complementary to the potential buyer's information. In the third example we study the bidding behavior in a second price, common value auction when bidders' private signals are complements or substitutes.

Many pairs of signals are neither complements nor substitutes if our definitions are used. This is because our definitions of these terms require certain conditions to be true in *all* decision problems. This is in the spirit of Blackwell's (1951) work. It seems plausible that more signals will satisfy the conditions for being substitutes or complements if we restrict attention to smaller classes of decision problems. In the context of Blackwell's original work this line of investigation has been taken up by Lehmann (1988), Persico (2000) and Athey and Levin (2001). The analogous research for our problem is left to a future paper.

Radner and Stiglitz (1984) consider a setting in which a one-dimensional real parameter indexes the quality of a signal. They show that the value of the signal in any decision problems is weakly increasing but not everywhere concave as the quality of information increases. In particular, a non-concavity occurs for any decision problem in the neighborhood of the parameter value for which the signal is entirely uninformative. Non-concavity of the value of a signal as the quality of the signal improves indicates increasing returns to scale in information. It may be possible to interpret an improvement in the quality of a signal as "making a further signal available", and one might be able to interpret a non-concavity of the value of information as a complementarity between an existing signal, and a further signal that might be made available. We have not yet explored whether we can make these analogies precise.

The idea that signals may be complements or substitutes has previously appeared in Sarvary and Parker (1997), who take consumers' valuation of signals, however, as exogenously given, and focus on competition among information providers. Complementarity and substitutability of signals has previously been referred to in an auction context by Milgrom and Weber

(1982b) and in a voting context complementarity of voters' information has been emphasized by Persico (2004). These papers consider very specific settings, however. Moreover, complementarity and substitutability of information in these papers is closely related to strategic interaction. In contrast, our paper is, to our knowledge, the first systematic study of substitutability and complementarity of signals in unipersonal decision problems.

The paper is organized as follows: Section 2 provides definitions. Section 3 contains our main completely general result. Section 4 studies in detail a symmetric binary example. Section 5 generalizes intuitive insights that we obtained for the symmetric binary example. Section 6 describes three economic applications. Section 7 concludes. Some of the proofs are contained in the appendix.

2 Definitions

The state of the world is a random variable \tilde{s} with realizations in a finite set S . Two signals are available: $\tilde{\sigma}_1$ which takes values in the finite set S_1 , and $\tilde{\sigma}_2$ which takes values in the finite set S_2 . We assume without loss of generality that $S_1 \cap S_2$ is empty. The joint distribution of signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ conditional on the state being equal to $s \in S$ is denoted by p_s . For $i = 1, 2$ the marginal distribution of signal $\tilde{\sigma}_i$ conditional on the state being equal to $s \in S$ is denoted by $p_{i,s}$.

Our objective in this section is to define when two signals are substitutes and when they are complements. We first need some auxiliary definitions.

Definition 1. A decision problem is a triple (π, A, u) where π is a probability distribution on S (the prior distribution), A is some finite set (the set of actions), and u is a function of the form: $u : A \times S \rightarrow \mathbb{R}$ (the utility function).

Definition 2. For given decision problem (π, A, u) :

- The value of not having any signal is:

$$V_\emptyset \equiv \max_{a \in A} \sum_{s \in S} (u(a, s) \pi(s)).$$

- For $i \in \{1, 2\}$ the value of having signal $\tilde{\sigma}_i$ alone is:

$$V_i \equiv \sum_{\sigma_i \in S_i} \max_{a \in A} \sum_{s \in S} (u(a, s) p_{i,s}(\sigma_i) \pi(s)).$$

- The value of having both signals is:

$$V_{1,2} \equiv \sum_{\sigma_1 \in S_1} \sum_{\sigma_2 \in S_2} \max_{a \in A} \sum_{s \in S} (u(a, s) p_s(\sigma_1, \sigma_2) \pi(s)).$$

We can now offer the two key definitions of this paper.

Definition 3. Signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are substitutes if for all decision problems (π, A, u) we have:³

$$V_1 - V_\emptyset \geq V_{1,2} - V_2$$

and

$$V_2 - V_\emptyset \geq V_{1,2} - V_1.$$

Definition 4. Signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are complements if for all decision problems (π, A, u) we have:

$$V_{1,2} - V_2 \geq V_1 - V_\emptyset$$

and

$$V_{1,2} - V_1 \geq V_2 - V_\emptyset.$$

The motivation for these definitions is best understood if one considers a setting in which an agent has to choose whether to purchase either one, or both, of the signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, and the agent's utility equals the expected utility from a decision problem of the type described in Definition 1 plus money holdings. Thus, the agent's utility is additively separable in the utility from the decision problem and money. In this case the utility differences $V_i - V_\emptyset$ and $V_{1,2} - V_j$ reflect the agent's willingness to pay for signal i if no signal is available (resp. if signal $j \neq i$ is available). Substitutability means that the willingness to pay for a signal decreases if the other signal becomes available, whereas complementarity means that the willingness to pay for a signal increases if the other signal becomes available.

The requirement that the inequalities in Definition 3 or 4 have to be true for *all* decision problems is very restrictive, and one may well ask whether any signal structures satisfy these requirements. We therefore give two simple examples.

³Note that the two inequalities in this definition, and also the two inequalities in Definition 4, are equivalent.

Example 1. States are: $S = \{+1, -1\}$. Signals take values in:⁴ $S_1 = S_2 = \{+1, -1\}$. The signal distributions are given by: $p_s(\sigma_1, \sigma_2) = 1/2 \Leftrightarrow \sigma_1 \cdot \sigma_2 = s$. Each individual signal's distribution is independent of the true state, yet together the two signals fully reveal the true state. Therefore, these signals are complements.

Example 2. States are: $S = \{+1, -1\}$. Signals take values in: $S_1 = S_2 = \{+1, -1\}$. The signal distributions are given by: $p_s(\sigma_1, \sigma_2) = 1 \Leftrightarrow \sigma_1 = \sigma_2 = s$. Each individual signal completely reveals the true state. Therefore, these signals are substitutes.

3 A General Result

To obtain a general characterization of signals that are complements or substitutes, we define two auxiliary signals, $\tilde{\sigma}_L$ and $\tilde{\sigma}_R$. Informally, the first of these signals, $\tilde{\sigma}_L$, can be described as follows. An unbiased coin is tossed. If “head” comes up, the decision maker is informed about the realization of $\tilde{\sigma}_1$. If “tails” comes up, the decision maker is informed about the realization of $\tilde{\sigma}_2$. Formally, the second auxiliary signal $\tilde{\sigma}_L$ has realizations in the set $S_L \equiv S_1 \cup S_2$.⁵ For given state $s \in S$, the probability that $\tilde{\sigma}_L$ has realization $\sigma_1 \in S_1$ is $p_{L,s}(\sigma_1) \equiv \frac{1}{2}p_{1,s}(\sigma_1)$, and the probability that $\tilde{\sigma}_L$ has realization $\sigma_2 \in S_2$ is $p_{L,s}(\sigma_2) \equiv \frac{1}{2}p_{2,s}(\sigma_2)$.

The second auxiliary signal, $\tilde{\sigma}_R$, is intuitively constructed as follows. An unbiased coin is tossed. If “head” comes up, the decision maker is informed about the realizations of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$. If “tails” comes up, the decision maker receives no information. Formally, the signal $\tilde{\sigma}_R$ has realizations in the set $S_L \equiv (S_1 \times S_2) \cup \{N\}$. Here, the symbol N denotes the case that the decision maker receives no information. For given state $s \in S$, the probability that $\tilde{\sigma}_R$ has realization $(\sigma_1, \sigma_2) \in S_1 \times S_2$ is $p_{R,s}(\sigma_1, \sigma_2) \equiv \frac{1}{2}p_s(\sigma_1, \sigma_2)$, and the probability that $\tilde{\sigma}_R$ has realization N is $p_{R,s}(N) \equiv \frac{1}{2}$.

Definition 5. For given decision problem (π, A, u) , and for $k \in \{L, R\}$, the value of having signal $\tilde{\sigma}_k$ is:

$$V_k \equiv \sum_{\sigma_k \in S_k} \max_{a \in A} \sum_{s \in S} (u(a, s) p_{k,s}(\sigma_k) \pi(s)).$$

⁴Examples 1 and 2 violate our assumption that $S_1 \cap S_2 = \emptyset$, but this is without consequence, and could be repaired by relabeling signals.

⁵Recall that we assume that $S_1 \cap S_2$ is empty.

Definition 6. Suppose $k, \ell \in \{L, R\}$ and $k \neq \ell$. Signal $\tilde{\sigma}_k$ is more valuable than signal $\tilde{\sigma}_\ell$ if for all decision problems (π, A, u) we have:

$$V_k - V_\emptyset \geq V_\ell - V_\emptyset.$$

Lemma 1. (i) Signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are substitutes if and only if signal $\tilde{\sigma}_L$ is more valuable than signal $\tilde{\sigma}_R$.

(ii) Signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are complements if and only if signal $\tilde{\sigma}_R$ is more valuable than signal $\tilde{\sigma}_L$.

Proof. For part (i) note that the two inequalities that define substitutes, $V_1 - V_\emptyset \geq V_{1,2} - V_2$ and $V_2 - V_\emptyset \geq V_{1,2} - V_1$ are equivalent to each other, and to: $\frac{1}{2}(V_1 + V_2) \geq \frac{1}{2}(V_{1,2} + V_\emptyset)$. But by definition the expression on the left hand side is the same as V_L , and the expression on the right hand side is the same as V_R . Thus (i) follows. The proof of part (ii) is analogous. \square

Blackwell (1951) has provided several necessary and sufficient conditions for one signal to be more valuable than another. Here, we quote one which has turned out to be useful in our context (for the proof of Proposition 2 below). We first need to introduce some notation and a definition. We state the notation in more general terms than we need for our next result because it will be useful later. Let π be a prior over the state space S . Using the joint conditional distribution of the two signals, we can then calculate the prior probability of observing signal realization $\sigma_k \in S_k$ (where $k \in \{1, 2\}$ or $k \in \{L, R\}$). We denote this probability by $q_\pi(\sigma_k)$. We can similarly define the prior probability $q_\pi(\sigma_1, \sigma_2)$ of observing the joint signal realization $(\sigma_1, \sigma_2) \in S_1 \times S_2$. Next, for $s \in S$, $k \in \{1, 2, L, R\}$, if $q_\pi(\sigma_k) > 0$ for $\sigma_k \in S_k$, we can define $q_\pi(s | \sigma_k)$ to be the posterior probability of s if σ_k is observed. If $q_\pi(\sigma_1, \sigma_2) > 0$ for $(\sigma_1, \sigma_2) \in S_1 \times S_2$, we can also define $q_\pi(s | \sigma_1, \sigma_2)$ to be the posterior probability of s if (σ_1, σ_2) is observed. Define the random variable $(q_\pi(s | \tilde{\sigma}_k))_{s \in S}$ to be the vector-valued random variable that describes the conditional probability of all states $s \in S$ where we condition on the realization $\tilde{\sigma}_k$. Finally, recall that a random variable \tilde{x} is a “mean-preserving spread” of another random variable \tilde{y} if $\tilde{x} = \tilde{y} + \tilde{z}$ where \tilde{z} is a random variable that has conditional expected value zero, conditional on each realization of \tilde{y} . Application of Blackwell’s (1951, Theorem 6) yields the following result:

Proposition 1. (i) Signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are substitutes if and only if $(q_\pi(s | \tilde{\sigma}_L))_{s \in S}$ is a mean-preserving spread of $(q_\pi(s | \tilde{\sigma}_R))_{s \in S}$. Here, π is the uniform prior over S .

(ii) Signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are complements if and only if $(q_\pi(s | \tilde{\sigma}_R))_{s \in S}$ is a mean-preserving spread of $(q_\pi(s | \tilde{\sigma}_L))_{s \in S}$. Here, π is the uniform prior over S .

4 A Symmetric Binary Example

We obtain results that have a much more immediate interpretation than the results of the previous section if we consider the special case that the state space and the signals are binary, and that the signals are symmetric. We shall show in this section that in this setting signals are substitutes if they are perfectly correlated, and they are complements if each signal can overturn in a sense that we shall make precise the “meaning” of the other signal. Example 1 in Section 2 is a case in which signals are perfectly correlated, and Example 2 in Section 2 is an example in which each signal can overturn the meaning of the other signal.

Formally, consider the special case that the state space and both signals are binary: $S = \{a, b\}$, $S_1 = \{\alpha, \beta\}$, and $S_2 = \{\hat{\alpha}, \hat{\beta}\}$. We also assume that the signals are symmetric, that is: $p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha})$ for all $s \in S$, which implies also: $p_{1,s}(\alpha) = p_{2,s}(\hat{\alpha})$ and $p_{1,s}(\beta) = p_{2,s}(\hat{\beta})$ for all $s \in S$.

Our next two assumptions rule out trivial cases. The first assumption is that for each realization of each signal there is strictly positive probability that it occurs in at least some state, i.e.: for every $i \in \{1, 2\}$ and every $\sigma_i \in S_i$ there is some $s \in S$ such that $p_{i,s}(\sigma_i) > 0$. The second assumption is that there is at least one informative signal realization: $p_a(\sigma_1, \sigma_2) \neq p_b(\sigma_1, \sigma_2)$ for at least one $(\sigma_1, \sigma_2) \in S_1 \times S_2$.

Our final assumption is without loss of generality. We assume that: $p_{1,a}(\alpha) \geq p_{1,b}(\alpha)$. Thus, if the decision maker receives signal $\tilde{\sigma}_1 = \alpha$ and no other signal, then his posterior probability of state a will not be lower than his prior probability. This implies that, if the decision maker receives signal $\tilde{\sigma}_1 = \beta$ and no other signal, then his posterior probability of state a is not higher than his prior probability. By symmetry, of course, the same is true for signals $\tilde{\sigma}_2 = \hat{\alpha}$ and $\tilde{\sigma}_2 = \hat{\beta}$.

We shall refer to the example defined in the above paragraphs as the “symmetric binary example”. Recall Examples 1 and 2 from Section 2. Both are special cases of the symmetric binary example. The following characterization shows that substitutes in the symmetric binary example share certain features with Example 2, and complements in the symmetric binary example share certain features with Example 1.

Proposition 2. *In the symmetric binary example: (i) Signals are substitutes if and only if:*

$$(C1) \quad p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0 \text{ for all } s \in S.$$

(ii) Signals are complements if and only if at least one of the following conditions holds:

$$(C2) \quad p_a(\alpha, \hat{\alpha}) \leq p_b(\alpha, \hat{\alpha});$$

$$(C3) \quad p_a(\beta, \hat{\beta}) \geq p_b(\beta, \hat{\beta}).$$

The proof of Proposition 2 in the Appendix uses Proposition 1. Here, we provide a discussion of this result. Condition (C1) says that the two signals are perfectly correlated, that is, if the decision maker knows the realization of one signal he can deduce with certainty what the realization of the other signal has been. Thus, in this case, in all decision problems $V_{1,2} - V_i = 0$ for $i \in \{1, 2\}$, while $V_i - V_\emptyset \geq 0$ for $i \in \{1, 2\}$, with strict inequality in most decision problems. It is obvious that signals are then substitutes, and hence the “if-part” of part (i) of Proposition 2 is trivial. Therefore, the proof of part (i) of Proposition 2 that is provided in the Appendix deals with the “only if-part” only.

Condition (C2) says that signal realization $(\alpha, \hat{\alpha})$ induces the decision maker to raise (or at least not to lower) his probability of state b in comparison to his prior probability for this state. This is despite of the fact that, as we have assumed, individually each of the signals $\tilde{\sigma}_1 = \alpha$ or $\tilde{\sigma}_2 = \hat{\alpha}$, if received alone without the other signal, induces the decision maker to raise (or at least not to lower) his subjective probability of state a . In other words, these signals’ meaning to the decision maker depends on whether they are received individually or together: Each signal alone is (weakly) indicative of state a , but if received together, they are (weakly) indicative of state b . Condition (C3) is the analogous statement for the signal realizations $(\beta, \hat{\beta})$.

To understand Proposition 2 more fully it helps to be more precise about the notion of the “meaning” of a signal realization $(\sigma_1, \sigma_2) \in S_1 \times S_2$. Let us say that “the meaning of (σ_1, σ_2) is s ” (where $s \in S$) if $p_s(\sigma_1, \sigma_2) > p_{s'}(\sigma_1, \sigma_2)$ (where $s' \neq s$), and that “the meaning of (σ_1, σ_2) is \emptyset ” if $p_s(\sigma_1, \sigma_2) = p_{s'}(\sigma_1, \sigma_2)$ (where $s' \neq s$). Thus, in the latter case the signal realization is uninformative.

For given distributions p_s we can construct a 2×2 matrix which describes the meaning of each signal realization. Figure 1 lists the seven different forms which this matrix can take under our assumptions. It is easily verified that each of these matrices can arise, and that no other matrix is compatible with our assumptions.

Proposition 2 now tells us that signals are complements if the signal structure is of the type labeled in Figure 1 as “Cases 3-6”. Signals are substitutes if the signal structure is of the type labeled in Figure 1 as “Case 7”, and if, moreover, the probability that uninformative signal realizations are observed is zero. Cases 1 and 2 in Figure 1 are neither complements nor substitutes. Note that Figure 1 makes clear that substitutes are non-generic in our framework, whereas complements are robust.

5 Generalizations

In this section we develop necessary (but not sufficient) conditions for signals to be substitutes or complements. The conditions that we present have a similar flavor as the characterizations that we obtained in Section 4 for the symmetric binary example.

Proposition 2 showed for the symmetric binary example that signals are substitutes if and only if each signal is redundant given the other signal. The following general result has a similar flavor.

Proposition 3. *If signals are substitutes, then for every prior π and every state s there is at least one signal i such that:*

$$q_\pi(s \mid \sigma_1, \sigma_2) \leq \max_{\sigma_i \in S_i} q_\pi(s \mid \sigma_i)$$

for every (σ_1, σ_2) that is observed with positive probability; and there is also some signal i such that:

$$q_\pi(s \mid \sigma_1, \sigma_2) \geq \min_{\sigma_i \in S_i} q_\pi(s \mid \sigma_i).$$

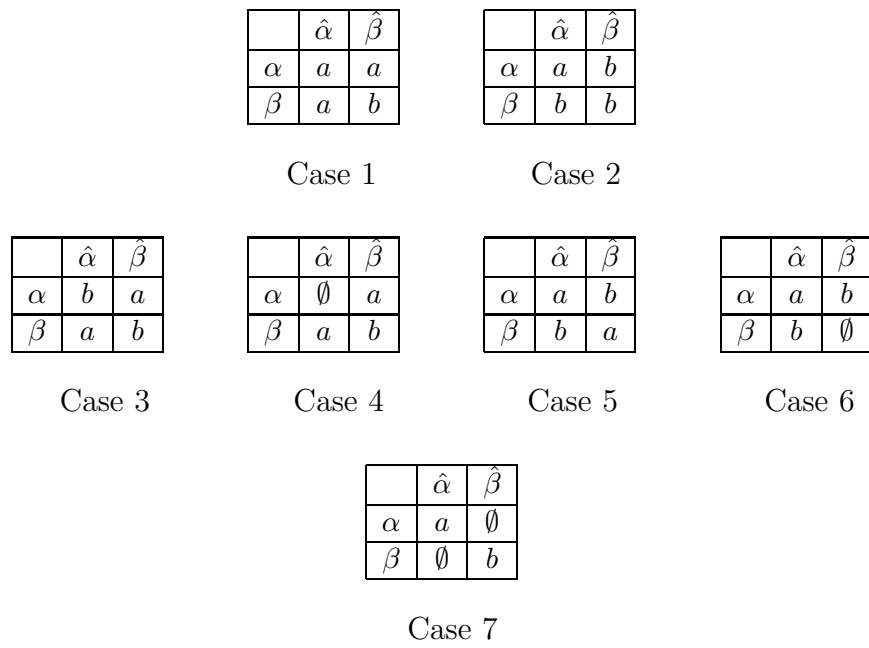


Figure 1: Possible meanings of signal realizations in the symmetric binary example

for every (σ_1, σ_2) that is observed with positive probability.

In words, the first inequality says that no joint signal realization (σ_1, σ_2) can provide stronger evidence in favor of state s than the strongest realization σ_i of signal i . Intuitively, thus, the signal $j \neq i$ provides no new information in favor of s whenever that realization of signal i is observed that provides strongest evidence of state s . The second inequality has an analogous interpretation.

Proposition 3, unlike Proposition 2, refers to prior distributions π . It would be interesting to obtain a formulation of Proposition 3 that refers to the conditional distribution of signals only. However, we have not been able to find such a formulation. The same remark applies to Proposition 4 below.

Proof. We only prove the first inequality. The proof of the second inequality is analogous. The proof is indirect. Suppose there are π and s such that for every i there exists a signal realization (σ_1, σ_2) that is observed with positive probability such that

$$q_\pi(s \mid \sigma_1, \sigma_2) > \max_{\sigma_i \in S_i} q_\pi(s \mid \sigma_i).$$

Then:

$$\max_{(\sigma_1, \sigma_2)} q_\pi(s \mid \sigma_1, \sigma_2) > \max_{\sigma_i \in S_i} q_\pi(s \mid \sigma_i)$$

where the maximum on the left hand side is taken over all signal realizations (σ_1, σ_2) that are observed with positive probability. Let \bar{q} denote a probability that is between the left hand side and the right hand side of the above inequality. Consider the decision problem with the prior π , action set $A = \{T, B\}$, and payoff function given by $u(T, s) = 1 - \bar{q}$, $u(T, s') = 0$ if $s' \neq s$, $u(B, s) = 0$, and $u(B, s') = \bar{q}$ if $s' \neq s$. The decision maker will choose T if and only if his posterior probability of state s is at least \bar{q} . The prior probability $\pi(s)$ is not more than \bar{q} because, for any i , it is a convex combination of $q_\pi(s \mid \sigma_i)$ for $\sigma_i \in S_i$. Therefore, without information, the agent chooses B . By the definition of \bar{q} , no signal realization $\sigma_i \in S_i$ of any signal i will give the decision maker an incentive to switch to T . Therefore, $V_i = 0$ for $i = 1, 2$. However, if the joint signal realization (σ_1, σ_2) is observed, the decision maker switches to T . Because this signal realization has positive prior probability this implies: $V_{1,2} > 0$. It then follows that the signals are not substitutes. \square

Proposition 2 showed that signals are complements in the symmetric binary example if and only if the meaning of each signal realization can be reversed by the realization of the other signal. Proposition 4 is a general result that has a somewhat similar flavor.

Proposition 4. *Suppose signals are complements, and suppose there is a prior π , a state s , a signal i , and a realization σ_i of signal i that is observed with positive probability, such that*

$$q_\pi(s \mid \sigma_i) \neq \pi(s).$$

Then there is at least one realization σ_j of signal $j \neq i$ that is observed with positive probability, and at least one realization σ'_i of signal i such that (σ_j, σ'_i) is observed with positive probability, such that one of the following holds:

- $q_\pi(s \mid \sigma_j) > \pi(s)$ and $q_\pi(s \mid \sigma_j, \sigma'_i) < \pi(s)$
- $q_\pi(s \mid \sigma_j) < \pi(s)$ and $q_\pi(s \mid \sigma_j, \sigma'_i) > \pi(s)$
- $q_\pi(s \mid \sigma_j) = \pi(s)$ and $q_\pi(s \mid \sigma_j, \sigma'_i) \neq \pi(s)$

In words, the condition on which the Proposition is based says that at least one realization of signal i alone changes the decision maker's belief that the true state is s . The result is thus based on a weak condition that ensures that signal i is informative. The first and the second bullet points then say that there are realizations of signal j alone, and of signal i and j together, such that the realization of signal i reverses the meaning of the realization of signal j . The third bullet point says that there is a realization of signal j that leaves the prior unchanged, but if the realization of signal i is observed, the prior does change.

Proof. Let π , s , and i be as described in the Proposition. Consider the decision problem with prior π , action set $A = \{T, B\}$, and payoff function given by $u(T, s) = 1 - \pi(s)$, $u(T, s') = 0$ if $s' \neq s$, $u(B, s) = 0$, $u(B, s') = \pi(s)$ if $s' \neq s$. The decision maker will choose T if and only if his posterior probability of state s is at least $\pi(s)$. Hence, without information, the agent is willing to choose T . By assumption, since $q_\pi(s \mid \sigma_i) \neq \pi$ for some σ_i , there is some realization σ'_i of signal i which induces the agent to switch to action B . Hence signal i has strictly positive value: $V_i - V_\emptyset > 0$.

Now suppose that, contrary to the assertion, $q_\pi(s \mid \sigma_j) > \pi(s)$ implied $q_\pi(s \mid \sigma_j, \sigma'_i) \geq \pi(s)$ for all σ'_i such that (σ_j, σ'_i) is observed with positive

probability, $q_\pi(s | \sigma_j) < \pi(s)$ implied $q_\pi(s | \sigma_j, \sigma'_i) \leq \pi(s)$ for all σ'_i such that (σ_j, σ'_i) is observed with positive probability, and $q_\pi(s | \sigma_j) = \pi(s)$ implied $q_\pi(s | \sigma_j, \sigma'_i) = \pi(s)$ for all σ'_i such that (σ_j, σ'_i) is observed with positive probability. Then no realization σ_i that the agent observes in addition to any realization σ_j induces a strict preference for changing actions. Therefore, signal i has no additional value when signal j is available: $V_{1,2} - V_j = 0$. But this, together with $V_i - V_\emptyset > 0$, implies that signals are not complements. \square

We now explore examples in which signals are complements, yet there is no reversal of the meaning of the signals, i.e. cases in which the third bullet point in Proposition 4 is satisfied. We distinguish two cases. Firstly, it may be that the third bullet point in Proposition 4 holds for all signal realizations σ_j and hence signal j by itself is uninformative. It turns out that in this case signals are always complements.

Proposition 5. *If there is a signal j such that the marginal distribution of signal j does not depend on the state, i.e. such that*

$$p_{j,s} = p_{j,s'} \text{ for all } s, s' \in S$$

then signals are complements.

Proof. Because signal j is uninformative we have: $V_j = V_\emptyset$ in all decision problems. Because $V_{1,2} - V_i \geq 0$ in all decision problems, the condition defining complements is satisfied. \square

One trivial case in which Proposition 5 applies is, of course, the case in which signal j is of no value when combined with signal i . However, there are other cases, as the following example shows.

Example 3. *There are two states: $S = \{a, b\}$. Signal 1 has two realizations: $\{\alpha, \beta\}$, and signal 2 has two realizations: $\{H, L\}$. Let $1/2 \leq q \leq p \leq 1$ and let $r \in [0, 1]$. The joint distributions of the signals conditional on the state are given in Figure 2.*

In Example 3 signal 2 indicates the precision of signal 1, but does not in itself contain information about the true state. Signal 1's precision is high (H) with probability r , and low with probability $1 - r$. When signal 1's precision is high, it indicates the true state with probability p . If signal 2's precision is low, it indicates the true state with probability q .

	H	L
α	rp	$(1-r)q$
β	$r(1-p)$	$(1-r)(1-q)$

	H	L
α	$r(1-p)$	$(1-r)(1-q)$
β	rp	$(1-r)q$

State a
State b

Figure 2: Conditional Signal Distributions in Example 3

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
α	0	0	p
β	0	0	0
γ	$1-p$	0	0

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
α	0	0	0
β	0	0	p
γ	0	$1-p$	0

State a
State b

Figure 3: Conditional Signal Distributions in Example 4

Because signal 2 is independent of the true state, its value by itself is zero: $V_2 - V_\emptyset = 0$. Moreover, it will always be true that $V_{1,2} - V_1 \geq 0$. Therefore, signals 1 and 2 will be weak complements. For some decision problems, however, signal 1 will be useful only if it shifts player 1's prior sufficiently strongly, and this will only be possible if the decision maker knows signal 1's precision to be high. In such decision problems: $V_{1,2} - V_1 > 0$. In such problems, signals 1 and 2 will be strict complements.

The second case of complementary signals with no signal reversion is the case in which the third bullet point in Proposition 4 is satisfied for some, but not all signal realizations σ_j . In this case, signal j by itself does potentially contain useful information. We now give an example of this case.

Example 4. *There are two states: $S = \{a, b\}$. Signal 1 has three realizations: $\{\alpha, \beta, \gamma\}$, and signal 2 has three realizations: $\{\hat{\alpha}, \hat{\beta}, \hat{\gamma}\}$. Let $p \in (0, 1)$. The joint distributions of the signals conditional on the state are given in Figure 3.*

Suppose the decision maker in Example 4 observes only the realization of signal σ_1 . If the decision maker observes that this realization is α she is certain that the state is a . Similarly, if she observes that the realization of signal σ_1 is β , then she is certain that the state is b . These two signal realizations are completely revealing about the state. However, if the decision maker observes signal realization γ , then her prior about the state is

unchanged. Analogous statements hold for $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$. Finally, considering the joint distribution of the signals, note that each signal is informative if and only if the other signal is uninformative. As a consequence, the marginal value of each signal is independent of whether the other signal is acquired, and both signals are complements. The condition in Proposition 4 is satisfied although the meaning of no signal is reversed, because each signal has realizations that leave the prior unchanged yet the realization of the other signal may move the prior, and thus the third bullet point in Proposition 4 holds.

Signals in Example 4 are also substitutes. The conditions of Proposition 3 are satisfied because each signal has *some* realization which indicates the true state with certainty. Thus, no realization of the other signal can raise the decision maker's probability of that state, nor can any realization of the other signal lower the decision maker's probability of the other state. Nevertheless, *ex ante*, each signal has positive marginal value, even if the other signal has been acquired, because each signal has *some* realization which does not indicate the true state with certainty. As part (i) of Proposition 2 showed, in the symmetric binary example if signals are substitutes each signal has zero marginal value if the other signal has been acquired. Example 4 demonstrates how, in a more general setting, signals can be substitutes without being perfectly correlated.

6 Economic Applications

In this section, we explore implications of substitutability and complementarity in three simple economic examples. When studying examples, we somewhat change the perspective of the previous sections, as in each example we deal with a specific set of decision problems rather than with all decision problems.

6.1 Strategic Information Acquisition

Consider two players $i = 1, 2$ who play a game. Before playing the game, the players decide independently and simultaneously whether or not to acquire signal $\tilde{\sigma}_i$ at cost $c_i \geq 0$, and then players meet and share the information acquired.

We are interested in how the players' information acquisition incentives are affected by the fact that signals are complements or substitutes. To address this question, we focus on a simple class of games in which each player chooses an action $a_i \in A_i$, and for $i \neq j$ player i 's (state-dependent) utility function is of the form

$$u^i(a_1, a_2, s) = w^i(a_i, s) + \lambda w^j(a_j, s),$$

where w^i, w^j are (non-strategic) utility functions, and λ is a real number. If $\lambda \geq 0$, there is a positive spill-over from player j to player i , and if $\lambda < 0$, there is a negative spill-over from player j to player i .

This utility specification allows us to apply the notions of complementarity and substitutability developed in the previous parts of the paper. This is so since with additive utility a player's value of information is independent of the other player's action.⁶

Denote by $V_{1,2}^i$ (resp. $V_i^i, V_j^i, V_\emptyset^i$) the value to player i of having both signals (resp. signal i , signal j , no signal) in the decision problem (π, A_i, w^i) . Hence player i 's equilibrium utility in the second stage is $V^i + \lambda V^j$ given the information available.

Let $\Delta_i(k)$ be player i 's information acquisition incentive when player j acquires $k = 0, 1$ signals:

$$\begin{aligned} \Delta_i(0) &= (V_i^i - V_\emptyset^i) + \lambda(V_i^j - V_\emptyset^j) - c_i, \\ \Delta_i(1) &= (V_{1,2}^i - V_j^i) + \lambda(V_{1,2}^j - V_j^j) - c_i. \end{aligned}$$

The information acquisition game displays strategic complements (resp. substitutes) if and only if $\Delta_i(1) - \Delta_i(0) \geq 0$ (resp. $\Delta_i(1) - \Delta_i(0) \leq 0$). We have:

$$\Delta_i(1) - \Delta_i(0) = [(V_{1,2}^i - V_j^i) - (V_i^i - V_\emptyset^i)] + \lambda[(V_{1,2}^j - V_j^j) - (V_i^j - V_\emptyset^j)].$$

Consider the case in which signals are complements. Then the differences in the square brackets are non-negative. Suppose they are strictly positive.

⁶In general games, when information acquisition is public, having or not having a signal will trigger a different response by the other player. Hence, a player faces a different decision problem depending on whether he has or does not have a signal. In such cases our notions of complementarity and substitutability are not applicable in a straightforward manner.

Then if there are positive spill-overs ($\lambda \geq 0$), there are strategic complementarities in information acquisition. If there are sufficiently large negative spill-overs ($\lambda \ll 0$), there are strategic substitutes in information acquisition.

It is easy to see the reason for this. The information acquired becomes also available to the other player. In the presence of positive spill-overs, a player benefits from the other player having more information. Accordingly, the additional value of an additional signal is reinforced. In the presence of negative spill-overs, a player is harmed from the other player having more information, and the additional value of an additional signal is accordingly reduced.

When signals are substitutes, the differences in the square brackets are non-positive, and the reverse statements hold. We conclude that strategic complementarity (subst.) in information acquisition is not only a matter of informational complementarity (subst.) but depends also on the nature of the game.

To analyze equilibrium information acquisition, we consider the symmetric case in which $w^i = w^j = w$ and where signal 1 and signal 2 have the same value: $V_1^i = V_2^i$ (from now on, we drop superindices). We also assume that c_i is player i 's private information, and let c_1 and c_2 be independently and uniformly distributed on $[\underline{c}, \underline{c} + 1]$ for $\underline{c} \geq 0$. These restrictions permit simple graphical illustrations. We are interested in comparative statics properties of the equilibrium of the information acquisition game with respect to λ . It turns out that these properties depend crucially on whether signals are complements or substitutes.

We focus on symmetric equilibria in which each player i acquires information if his cost type c_i is smaller than a threshold $\hat{c} \in [\underline{c}, \underline{c} + 1]$. Let $\theta = \hat{c} - \underline{c}$ be the ex ante probability with which a player acquires information under such a threshold strategy. With abuse of notation, we refer to the equilibrium information acquisition probability θ^* itself as an equilibrium.

Given a player acquires information with ex ante probability θ , the other player's expected gain from acquiring information if his cost type is c is given as

$$\begin{aligned} \Delta(\theta, c) &= \theta(1 + \lambda)(V_{1,2} - V_2) + (1 - \theta)(1 + \lambda)(V_1 - V_\emptyset) - c \\ &= (1 + \lambda)[(V_{1,2} - V_2) - (V_1 - V_\emptyset)]\theta + (1 + \lambda)(V_1 - V_\emptyset) - c \\ &= (1 + \lambda)\delta\theta + (1 + \lambda)(V_1 - V_\emptyset) - c, \end{aligned}$$

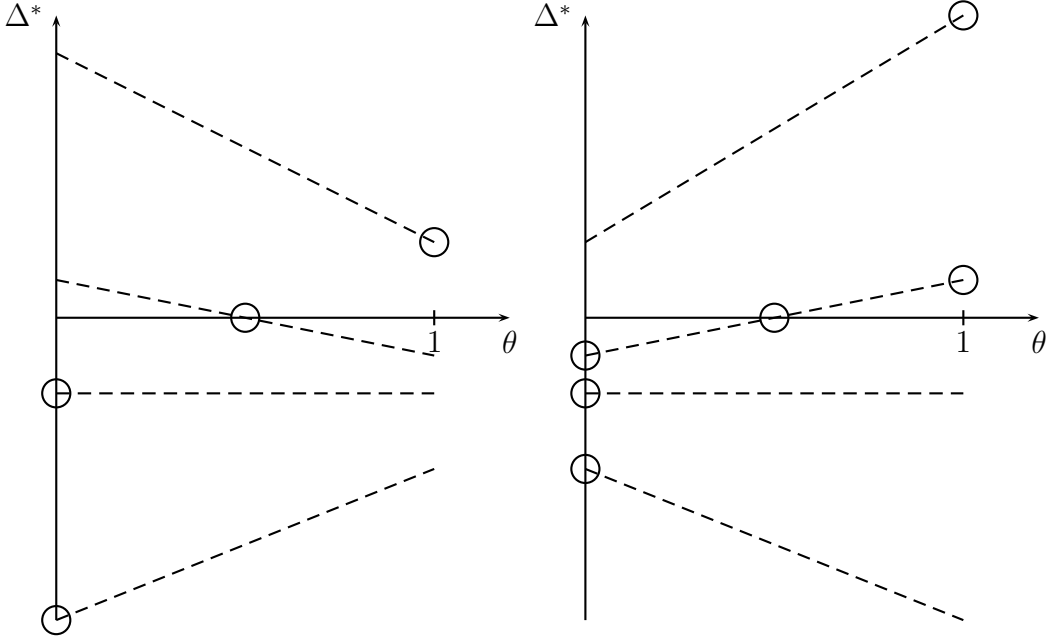


Figure 4: Information acquisition equilibria when signals are substitutes (left) and complements with large \underline{c} (right).

where we define $\delta = (V_{1,2} - V_2) - (V_1 - V_\emptyset)$. An interior equilibrium θ^* is given by the mass of cost types below the cost type who is just indifferent, given θ^* , i.e. θ^* solves $\Delta^*(\theta^*) = 0$ where

$$\Delta^*(\theta) \equiv \Delta(\theta, \underline{c} + \theta) = [(1 + \lambda)\delta - 1]\theta + (1 + \lambda)(V_1 - V_\emptyset) - \underline{c}.$$

Furthermore, if $\Delta^*(1) \geq 0$, then even the highest cost type $\underline{c} + 1$ has a positive information acquisition incentive given the other player acquires information with probability 1, hence $\theta^* = 1$ is then an equilibrium. Likewise, if $\Delta^*(0) \leq 0$, then $\theta^* = 0$ is an equilibrium.

Next, we illustrate these observations graphically. Figure 4 displays the function Δ^* as λ varies. Note that the slope of Δ^* is positive if and only if $(1 + \lambda)\delta > 1$ and that $\Delta^*(1)$ is increasing in λ . Equilibria are indicated by circles.

Consider first the case in which signals are substitutes, i.e. $\delta < 0$ (left panel). For $\lambda < (1 - \delta)/\delta$, the slope of Δ^* is positive (bottom dashed line). Since $\Delta^*(1) < 0$ in this case, the unique equilibrium features no information acquisition. For $\lambda = (1 - \delta)/\delta$, the slope of Δ^* is zero, and $\Delta^*(1) < 0$ (second

to bottom dashed line). Hence, there is still no information acquisition in equilibrium. As λ increases, the slope becomes negative and Δ^* increases. At some point, there is a unique interior equilibrium (second to top dashed line). For large λ , the unique equilibrium has both players acquiring information (top dashed line).

Suppose now that signals are complements, i.e. $\delta > 0$ (right panel). It turns out that a qualitatively different picture to the case $\delta < 0$ arises if \underline{c} is sufficiently large.⁷ For $\lambda < (1 - \delta)/\delta$, the slope of Δ^* is now negative (bottom dashed line). If \underline{c} is large, then $\Delta^*(1) < 0$, and the unique equilibrium features no information acquisition. For $\lambda = (1 - \delta)/\delta$, the slope of Δ^* is zero (second to bottom dashed line), and there is still no information acquisition in equilibrium if \underline{c} is large. As λ increases, the slope becomes larger and Δ^* increases. At some point, there are multiple equilibria (second to top dashed line). Here, even the lowest cost type would not acquire information in the absence of the other player. Thus, if each player believes the other player to abstain from information acquisition, there will be no information acquisition in equilibrium. However, because signals are complements and spillovers are large, there are strategic complementarities in information acquisition. Thus, a player's information acquisition is stimulated if he believes that the other player acquires information. Thus, everyone acquiring information is an equilibrium. When λ becomes large, the unique equilibrium has any player acquiring information (top line).

In Figure 5, we trace the equilibrium information acquisition probability θ^* as λ increases. The left panel displays the case when signals are substitutes. In this case, we have a smooth upward sloping “demand” for information. As spillovers λ go up, the likelihood of acquiring information goes up smoothly. This differs markedly from the right panel where signals are complements and \underline{c} is sufficiently large. Here, small changes in spillovers might lead to a sharp drop or rise in information acquisition. Note that it appears as if along the interior equilibrium, demand for information could go down as spillovers increase.

6.2 Information Revelation By An Informed Seller

In this subsection, we revisit results regarding a monopolist's incentives to reveal his private information about the quality of the good that he is selling.

⁷For the low cost case the picture is qualitatively similar to the case $\delta < 0$.

- Public information: the realization $\tilde{\sigma}_2$ is common knowledge.

The comparison of the seller's expected revenue in the two scenarios characterizes the seller's incentive to reveal publicly a signal when the seller does not observe the signal unless it is made public. An alternative, perhaps more realistic, scenario is when the seller privately observes some non-verifiable information. This, however, introduces further complications since in equilibrium the buyer forms beliefs about the seller's private information that must be consistent with the seller's pricing.

We assume that the seller makes a take-it or leave-it price offer to the buyer. Of course, this offer may depend on the realization of $\tilde{\sigma}_2$ in the public information case but not in the no information case. It may be shown that in this framework the seller cannot do better than using a posted price.

We first recapitulate Theorem 18 in Milgrom and Weber (1982a), adapted to our example. The result is based on the assumption that the random variables $(\tilde{s}, \tilde{\sigma}_1, \tilde{\sigma}_2)$ are affiliated and that the buyer's utility function is increasing in the state. Therefore, naturally, we define b to be larger than a . Signals can then only be affiliated with the state if we define β (resp. $\hat{\beta}$) to be larger than α (resp. $\hat{\alpha}$) because the former signals, by assumption, induce a larger posterior probability of b than the latter signals. With this ordering of signals the affiliation condition is now a well-defined condition (see Milgrom and Weber (1982a) and Ottaviani and Pratt (2001) for the particular case of discrete random variables).

Proposition 6. *Suppose the random variables $(\tilde{s}, \tilde{\sigma}_1, \tilde{\sigma}_2)$ are affiliated. Then the maximum expected revenue of the seller is greater under public information than under no information.*

Affiliation of the signals with each other and with the state implies in our model that observing $(\alpha, \hat{\alpha})$ induces a posterior probability of b that is no larger than the posterior probability of b induced by observing just α , and that observing $(\beta, \hat{\beta})$ induces a posterior probability of b that is no less than the posterior probability of b induced by observing just β .⁹ Thus, by Proposition 2, affiliation and complementarity of signals are not compatible.

⁹Note that the affiliation inequality $p_b(\alpha, \hat{\alpha}) p_a(\alpha, \hat{\beta}) \leq p_a(\alpha, \hat{\alpha}) p_b(\alpha, \hat{\beta})$ is equivalent to $\frac{p_b(\alpha, \hat{\alpha})}{p_a(\alpha, \hat{\alpha})} \leq \frac{p_b(\alpha)}{p_a(\alpha)} \leq \frac{p_b(\alpha, \hat{\beta})}{p_a(\alpha, \hat{\beta})}$, and similarly, the affiliation inequality $p_b(\beta, \hat{\alpha}) p_a(\beta, \hat{\beta}) \leq p_a(\beta, \hat{\alpha}) p_b(\beta, \hat{\beta})$ is equivalent to $\frac{p_b(\beta, \hat{\alpha})}{p_a(\beta, \hat{\alpha})} \leq \frac{p_b(\beta)}{p_a(\beta)} \leq \frac{p_b(\beta, \hat{\beta})}{p_a(\beta, \hat{\beta})}$.

We now demonstrate that Proposition 6 need not hold with complementary signals and when m is sufficiently large. We focus on Cases 3 and 5 of Figure 1, ruling out the knife-edge Cases 4 and 6.

Proposition 7. *In Case 3, the maximum expected revenue of the seller is greater under no information than under public information if m is large enough. In Case 5, the same statement holds if in addition it holds that $q_\pi(b \mid \alpha) \geq q_\pi(b \mid \beta, \hat{\beta})$.*

Proof. The seller's expected revenue is equal to the price multiplied by the probability of selling. Thus, if m is large enough, the seller achieves her maximum expected payoff by selling with probability one in both cases, under no information and under public information. Hence, the seller's optimal price equals the buyer's lowest possible valuation, given the seller's information. In the no information case, the optimal price is thus $m + q_\pi(b \mid \alpha)$.

Consider the public information case. In Case 3, when signal 2 is $\hat{\alpha}$, the buyer's lowest possible valuation is $m + q_\pi(b \mid \beta, \hat{\alpha})$, and when signal 2 is $\hat{\beta}$, the buyer's lowest possible valuation is $m + q_\pi(b \mid \alpha, \hat{\beta})$. Thus in either case, the seller achieves a payoff of $m + q_\pi(b \mid \beta, \hat{\alpha})$ which is clearly smaller than under the no information in Case 3.

In Case 5, when signal 2 is $\hat{\alpha}$, the buyer's lowest possible valuation is $m + q_\pi(b \mid \alpha, \hat{\alpha})$, and when signal 2 is $\hat{\beta}$, the buyer's lowest possible valuation is $m + q_\pi(b \mid \beta, \hat{\beta})$. In Case 5, if the condition stated in the proposition holds, this is in either case smaller than the price in the no information case. \square

To explain the difference in the result, we first note the two key features that drive Proposition 1. The first feature, which we already indicated above, is that affiliation implies posteriors to be well-ordered, that is, $q_\pi(b \mid \alpha) < q_\pi(b \mid \alpha, \hat{\beta}) < q_\pi(b \mid \beta)$. This allows the seller to replicate the no information selling outcome under public information in a particular way. For example if m is large, the seller optimally sets the price equal to $m + q_\pi(b \mid \alpha)$ under no information. Under public information, he can replicate this selling outcome by setting the price equal to $m + q_\pi(b \mid \alpha, \hat{\alpha})$ if the public signal is $\hat{\alpha}$ and equal to $m + q_\pi(b \mid \alpha, \hat{\beta})$ if the public signal is $\hat{\beta}$. Is this replication strategy profitable? To answer this, note that the replication strategy has a revenue equal to the expected price where the two prices are weighted with the *unconditional* distribution of signal 2. Moreover, if we take the expectation of these prices with respect to the *conditional* distribution

of signal 2, conditional on signal 1 taking on the low realization α , then we exactly obtain the no information price. Therefore, no information gives the seller a smaller profit if the conditional distribution of signal 2 is dominated in the first order sense by the unconditional distribution of signal 2. But this is precisely the second feature which holds under affiliation.

The difference between affiliation and complementarity that explains the difference between the results is that the first feature referred to in the previous paragraph is absent when signals are complements. Due to meaning reversals, posteriors are not well-ordered. As a consequence, the replication strategy of the previous paragraph does not replicate the no information outcome under public information when signals are complements.

6.3 A Second Price Auction

In this subsection we study the implications of complementarity and substitutability of signals if these signals are privately observed by bidders in a second price, common value auction. In particular, we show that complementarity of signals may imply that the second price, common value auction has no symmetric equilibrium in pure strategies, and that asymmetric equilibria involve strategies that are non-monotone in individual signals. The results that we find for the case of complementary signals contrast sharply with the properties of the second price auction in a setting with affiliated values (see Milgrom and Weber (1982a)).

We assume that a single indivisible good is sold through a second price auction to two bidders. Bidders submit their bids simultaneously. All non-negative real numbers are allowed as bids. The highest bidder wins the object. She pays the second highest bid. The second highest bidder wins nothing and pays nothing. Ties are resolved by tossing a fair coin.

We consider the symmetric binary example of Section 4. We assume that the state $s \in \{a, b\}$ represents the true value of the object and set $a = 0$ and $b = 1$. This value is common to both bidders. Bidder i has von Neumann-Morgenstern utility $s - p$ if she wins and pays a price p , and zero, if she does not win. To make the problem interesting, we assume that $\pi(b) \in (0, 1)$. Before submitting a bid, bidder i privately observes signal $\tilde{\sigma}_i$. We denote by $b_i(\sigma_i)$ bidder i 's bid when she observes $\sigma_i \in S_i$. We begin by studying symmetric Bayesian Nash equilibria in pure strategies of this game. Symmetry here means that $b_1(\alpha) = b_2(\hat{\alpha})$ and $b_1(\beta) = b_2(\hat{\beta})$.

By Proposition 2, if signals are substitutes, they are perfectly correlated. Thus, bidders' conditional expected values of the good are common knowledge among the bidders, and they are identical. Bidders engage in Bertrand-style competition for the good, and it is immediate that both bidders bidding the correct conditional expected value is then the only symmetric equilibrium in pure strategies.

Proposition 8. *If $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are substitutes, then there exists a unique symmetric equilibrium in pure strategies. It is: $b_1(\alpha) = q_\pi(b|\alpha, \hat{\alpha})$, and $b_1(\beta) = q_\pi(b|\beta, \hat{\beta})$.*

In the equilibrium described in Proposition 8 is formally the same as the equilibrium that Milgrom and Weber (1982a, Theorem 6) identified for the symmetric second price auction with affiliated signals. In the two bidder case each bidder bids the value that the good would have if the other bidder's signal were the same as the bidder's own signal. Of course, the equilibrium described in Proposition 8 implies that the price paid to the auctioneer is identical to the true value of the good, whereas in the general affiliated value model the winning bidder obtains a positive information rent.

Next, we consider the case that $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are complements. To simplify our exposition, we assume that $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ has full support conditional on each state, and that each signal is informative by itself: $p_a(\alpha) \neq p_b(\alpha)$. Moreover, we only consider Cases 3 and 5 of Figure 1. Recall from the previous subsection that complementarity and affiliation of signals are incompatible. We find that there is a sharp contrast between the case of affiliation and the case of complementarity, as the following result shows.

Proposition 9. *If $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ has full support conditional on each state, both signals are informative by themselves: $p_a(\alpha) \neq p_b(\alpha)$, and Cases 3 or 5 of Figure 1 apply, then there is no symmetric equilibrium in pure strategies.*

The proof of this result establishes first that any potential symmetric equilibrium in pure strategies would involve the same strategies as the equilibrium found in Milgrom and Weber (1982a, Theorem 6). The proof then shows that complementarity of signals implies that, in fact, these strategies are *not* a Bayesian equilibrium. Intuitively, the reason why a similar existence result does not apply in Cases 3 and 5 in our model is that in those cases the signals are not well-ordered. Whether α or β shifts the posterior upwards or downwards conditional on a realization σ_2 of the signal of the

other bidder depends on the value of σ_2 . By contrast, in Milgrom and Weber (1982a), signals are well-ordered. Their meaning is independent of the other agents' signals.

Proof. Any symmetric equilibrium in pure strategies would have to satisfy the following condition: if bidder 1 bids p when she observes σ_1 , the expected value of the good conditional on $\hat{\sigma}_1 = \sigma_1$ and conditional on bidder 2 bidding p when she uses the symmetric bid function must be equal to p . To see why, note that otherwise, bidder 1 would have incentives to deviate. For instance, suppose that the above conditional expected value is less than p . Then, bidder 1 gets negative expected utility when she observes σ_1 , bidder 2 bids p , both tie and bidder 1 wins. Thus, bidder 1 can improve by bidding $p - \varepsilon$ when she observes σ_1 for $\varepsilon > 0$ and ε small enough. This deviation only affects bidder 1's payoffs when bidder 1 ties in the above situation. A similar argument applies when bidder 1 obtains positive expected utility conditional on the event that she observes σ_1 , bidder 2 bids p , both tie and bidder 1 wins.

Next, note that in a symmetric equilibrium in pure strategies in which $b_1(\alpha) = b_1(\beta)$, the event that bidder 2 submits the same p does not provide information about her private signal. Thus, our necessary condition becomes $b_1(\sigma_1) = q_\pi(b|\sigma_1)$, for $\sigma_1 \in \{\alpha, \beta\}$, which together with $b_1(\alpha) = b_1(\beta)$ gives us a contradiction with our assumption that the signals are individually informative.

Consider next a symmetric equilibrium in pure strategies in which $b_1(\alpha) \neq b_1(\beta)$. Both bidders submit the same bid only if they observe the same signal realization. Thus, our necessary condition becomes $b_1(\alpha) = q_\pi(b|\alpha, \hat{\alpha})$ and $b_1(\beta) = q_\pi(b|\beta, \hat{\beta})$. We show next that if the bid function satisfies this condition, it must also satisfy that:

$$q_\pi(b|\beta, \hat{\alpha}) \in [\min\{b_1(\alpha), b_1(\beta)\}, \max\{b_1(\alpha), b_1(\beta)\}].$$

To see why, suppose, for instance, that $q_\pi(b|\beta, \hat{\alpha}) < \min\{b_1(\alpha), b_1(\beta)\}$. Then bidder 1 obtains zero expected utility if she and bidder 2 submit the higher bid. However, bidder 1 gets negative expected utility if she submits her higher bid and bidder 2 her lower bid. This is because the expected value of the good conditional on this event is equal to $q_\pi(b|\beta, \hat{\alpha})$, which by our starting assumption is less than bidder 2's bid. As a consequence, bidder 1 has an incentive to deviate and submit a bid that loses with probability one.

The argument for $q_\pi(b|\beta, \hat{\alpha}) > \max\{b_1(\alpha), b_1(\beta)\}$ is similar. In this case, bidder 1 has an incentive to replace her lower bid by a bid that allows her to win with probability one.

But now observe that the condition that we have derived in the previous paragraph is incompatible with complementary signals. In Case 3 of Figure 1 we have that $q_\pi(b|\beta, \hat{\alpha}) < \max\{b_1(\alpha), b_1(\beta)\}$, and in Case 5 of Figure 1 we have that $q_\pi(b|\beta, \hat{\alpha}) > \max\{b_1(\alpha), b_1(\beta)\}$. Thus, there cannot be a symmetric equilibrium in pure strategies. \square

While the second price auction does in general not have symmetric Bayesian equilibria in pure strategies when signals are complements, it does have asymmetric Bayesian equilibria, as our next result shows.

Proposition 10. *The following strategies constitute a Bayesian equilibrium:*

$$\begin{aligned} b_1(\alpha) &= \max\{q_\pi(b | \alpha, \hat{\alpha}), q_\pi(b | \alpha, \hat{\beta})\} \\ b_1(\beta) &= \max\{q_\pi(b | \beta, \hat{\alpha}), q_\pi(b | \beta, \hat{\beta})\} \\ b_2(\alpha) &= \min\{q_\pi(b | \alpha, \hat{\alpha}), q_\pi(b | \alpha, \hat{\beta})\} \\ b_2(\beta) &= \min\{q_\pi(b | \beta, \hat{\alpha}), q_\pi(b | \beta, \hat{\beta})\} \end{aligned}$$

Another Bayesian equilibrium can be found by swapping the roles of players 1 and 2 in this equilibrium.

We omit the proof of Proposition 10. It can easily be verified that ex post, after observing both players' signals, no player has an incentive to deviate. Thus, the strategies in Proposition 10 form not only a Bayesian, but an ex post equilibrium. The strategies in Proposition 10 are reminiscent of the strategies described in Theorem 6.3 in Milgrom (1981) for which Theorem 5.4.8 of Milgrom (2004) shows that they form an ex post equilibrium.

Note that Proposition 10 holds for all possible distributions of the signals in the symmetric binary example. If signals are affiliated, the equilibria of Proposition 10 coexist with the standard symmetric equilibrium in pure strategies. By contrast, in the case of complementary signals, there are only asymmetric equilibria in pure strategies. There may be other such equilibria than the one described in Proposition 10. Another difference between the case of affiliated signals and the case of complementary signals is that in the affiliated case the strategies described in Proposition 10 are monotonically increasing, that is: $b_i(\alpha) \leq b_i(\beta)$ for $i = 1, 2$. In Cases 3 and 5, by contrast,

the strategies described in Proposition 10 may be monotonically decreasing. This will be the case if $q_\pi(b \mid \alpha, \alpha) > q_\pi(b \mid \beta, \beta)$. In Case 3 the strategy of player 1, and in Case 5 the strategy of player 2 (as labeled in Proposition 10) will then be decreasing.

7 Conclusion

This paper has provided some insights into the nature of complementarity and substitutability relations among signals, but many questions remain open. Firstly, we have not been able to determine to which extent the intuitive insights of Section 4 generalize. Our results in Section 5 are only partial. Secondly, whereas in this paper we have sought characterizations that imply substitutability or complementarity in *all* decision problems, one might restrict attention to smaller classes of decision problems as they typically arise in economics, for example to monotone decision problems as in Athey and Levin (2001).

Complementarity and substitutability relations among signals may matter in economic contexts where agents hold private signals, and each agents' preferences depend on all signals, that is, in contexts with interdependent preferences. Such contexts arise naturally in auctions or in voting games. It seems worthwhile to explore the implications of complementarity and substitutability in those contexts.

Complementarity of signals may also matter when agents acquire signals sequentially. In this case, the second signal may be acquired when the agent already knows the realization of the first signal. By contrast, in our setting, each signal is acquired without knowing the realization of the other signal. Extending our results to a setting where agents evaluate signals knowing the realization of other signals is another project for future work.

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Appendix

Proof of Proposition 2

We shall work with the notation for posteriors introduced in Section 3. Note that our assumption that for every signal realization there is a state in which this signal realization is observed with positive probability implies that the conditional probabilities $q_\pi(s | \sigma_i)$ are all well-defined for all $\sigma_i \in S_i$.

Part (i): As mentioned in the main text, the “if-part” of part (i) of Proposition 2 is obvious. Therefore, we focus on the “only if-part”. We proceed in two steps.

Step 1: We show that substitutability of the two signals implies that there is a $\lambda \geq 0$ such that:

$$(p_b(\alpha, \hat{\alpha}), p_b(\alpha, \hat{\beta})) = \lambda(p_a(\alpha, \hat{\alpha}), p_a(\alpha, \hat{\beta})). \quad (1)$$

Intuitively, this equality says that when the decision maker receives signal $\sigma_1 = \alpha$, then his or her beliefs will be the same when the second signal is $\sigma_2 = \hat{\alpha}$ and when it is $\sigma_2 = \hat{\beta}$. Before we proceed to the proof, we note that if (1) is true, symmetry implies:

$$(p_b(\alpha, \hat{\alpha}), p_b(\beta, \hat{\alpha})) = \lambda(p_a(\alpha, \hat{\alpha}), p_a(\beta, \hat{\alpha})). \quad (2)$$

Moreover, arguments analogous to those that we use below to show (1) also prove that there is a $\gamma \geq 0$ such that:

$$(p_b(\beta, \hat{\alpha}), p_b(\beta, \hat{\beta})) = \gamma(p_a(\beta, \hat{\alpha}), p_a(\beta, \hat{\beta})); \quad (3)$$

$$(p_b(\alpha, \hat{\beta}), p_b(\beta, \hat{\beta})) = \gamma(p_a(\alpha, \hat{\beta}), p_a(\beta, \hat{\beta})). \quad (4)$$

To prove (1) we now distinguish two cases.

CASE 1: Suppose that there is only one $\sigma_2 \in S_2$ such that $p_a(\alpha, \sigma_2) > 0$. Our proof of the assertion is indirect. The condition which we seek to prove can only be violated if $p_b(\alpha, \sigma'_2) > 0$ for $\sigma'_2 \neq \sigma_2$. Consider the prior π given by: $\pi(a) = \pi(b) = 0.5$. The posterior probability of state a after receiving signal α is less than the posterior probability of state a after receiving signal (α, σ_2) :

$$q_\pi(a | \alpha) < q_\pi(a | \alpha, \sigma_2).$$

	a	b
T	$1 - x$	0
B	0	x

Figure 6: A payoff function

Pick some number x strictly between these two posterior probabilities, and consider the following decision problem. The set of actions is: $A = \{T, B\}$. The payoff function u is defined in Figure 6.

Observe that the decision maker will choose action T if and only if his posterior probability of state a is at least x . Now suppose $x > 0.5$. Then, without any information, the decision maker will choose B . Now suppose the decision maker had only signal $\tilde{\sigma}_1$ available. No realization of signal 1 can convince the decision maker to switch to action T . Indeed, if $\tilde{\sigma}_1 = \beta$, then the decision maker revises his prior probability of state a (weakly) downwards. Thus, he will choose B . On the other hand, if $\tilde{\sigma}_1 = \alpha$, then by construction of x the posterior probability of a is below x . Again, the decision maker will choose B . Because no realization of $\tilde{\sigma}_1$ can convince the decision maker to change his choice in comparison to the case in which he receives no signal, we have: $V_1 - V_\emptyset = 0$.

On the other hand, if the decision maker receives two signals, then for some realizations of the signal he changes his action. Specifically, if the decision maker observes (α, σ_2) , then the probability which he attaches to a rises above x , and he will choose T . Therefore, with positive probability, two signals together will have a realization that induces the decision maker to change his action. This implies: $V_{1,2} - V_\emptyset > 0$. Because: $V_{1,2} - V_1 > V_1 - V_\emptyset$, we have obtained a contradiction to substitutability.

CASE 2: We have $p_a(\alpha, \hat{\alpha}) > 0$ and $p_a(\alpha, \hat{\beta}) > 0$. We prove our assertion indirectly. Our assertion is violated if and only if:

$$\frac{p_b(\alpha, \hat{\alpha})}{p_a(\alpha, \hat{\alpha})} \neq \frac{p_b(\alpha, \hat{\beta})}{p_a(\alpha, \hat{\beta})}.$$

Suppose the left hand side were smaller than the right hand side. Then signal $(\alpha, \hat{\alpha})$ raises the posterior probability of state a to a higher level than signal $\sigma_1 = \alpha$ alone. Thus, we can repeat the construction displayed in Case 1 and

obtain a contradiction to substitutability. The same argument works if the right hand side is smaller than the left hand side.

Step 2: We now deduce from equations (1)-(4) that Condition (C1) in Proposition 2 must hold. Suppose that $p_a(\alpha, \hat{\beta}) > 0$. Then we would have $\lambda = \gamma$ because by (1):

$$p_b(\alpha, \hat{\beta}) = \lambda p_a(\alpha, \hat{\beta}),$$

and by (4):

$$p_b(\alpha, \hat{\beta}) = \gamma p_a(\alpha, \hat{\beta}).$$

Thus, we would conclude from (1)-(4) that $p_a(\sigma_1, \sigma_2) = p_b(\sigma_1, \sigma_2)$ for all $(\sigma_1, \sigma_2) \in S_1 \times S_2$. But this contradicts our assumption that at least some signal realization is informative. We can thus conclude that $p_a(\alpha, \hat{\beta}) = 0$. By symmetry: $p_a(\beta, \hat{\alpha}) = 0$. By equations (1) and (2) this implies: $p_b(\alpha, \hat{\beta}) = p_b(\beta, \hat{\alpha}) = 0$.

Part (ii): We begin by showing that the assertion is rather trivially true if $p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0$ for both $s \in \{a, b\}$, i.e. condition (C1) holds. The trivial way in which the assertion is true is that if this condition is true, signals cannot be complements, nor can conditions (C2) or condition (C3) be satisfied.

Lemma 2. *Suppose $p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0$ for $s \in \{a, b\}$. Then the signals are not complements. Moreover, neither condition (C2) nor condition (C3) holds.*

Proof. Suppose $p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0$ for $s \in \{a, b\}$. Then signals are perfectly correlated. As explained in the main text, this implies that the value of a second signal, if one signal has already been observed, is zero. Moreover, because by assumption at least some signal realization is informative, the value of one signal is in some decision problems positive. Therefore, the signals cannot be complements.

Now suppose either (C2) or (C3) were true. By assumption $p_{i,a}(\alpha) \geq p_{i,b}(\alpha)$ and $p_{i,a}(\beta) \leq p_{i,b}(\beta)$. Because at least one of (C2) and (C3) holds, one of these inequalities has to hold with equality. Because probabilities add up to one, both inequalities then hold with equality. This implies that all signal realizations occur with the same probability in both states, and hence that all signals are uninformative. We have ruled this out by assumption. Thus we have obtained a contradiction. \square

Lemmas 2 implies that it is sufficient to prove part (ii) of Proposition 2 for the case that there is at least one state s such that $p_s(\alpha, \hat{\beta}) > 0$. We shall make this assumption from now on without further mentioning.

Recall the definition of the signals $\tilde{\sigma}_L$ and $\tilde{\sigma}_R$ in Section 3. By Proposition 1, we can show the claim of part (ii) of Proposition 2 by showing for the uniform prior over the state space that the distribution of the decision maker's posterior probabilities over the state space S after observing $\tilde{\sigma}_R$ is a mean-preserving spread of the distribution of the decision maker's posterior probabilities under $\tilde{\sigma}_L$ if and only if condition (C2) or (C3) holds. Since the state space is binary, the posterior $(q_\pi(s | \tilde{\sigma}_R))_{s \in \{a, b\}}$ is a mean preserving spread of the posterior $(q_\pi(s | \tilde{\sigma}_L))_{s \in \{a, b\}}$ if and only if the mean preserving spread relation between $q_\pi(s | \tilde{\sigma}_R)$ and $q_\pi(s | \tilde{\sigma}_L)$ holds for one of the components $s \in \{a, b\}$. We shall work with the posterior probability of b given $\tilde{\sigma}_k$. For $k \in \{L, R\}$ denote by F_k the cumulative distribution function of $q_\pi(b | \tilde{\sigma}_k)$, and define $H_k(x) = \int_0^x F_k(q) dq$ for all $x \in [0, 1]$. By Theorem 2 of Rothschild and Stiglitz (1970) the distribution F_R is a mean-preserving spread of F_L if and only if

$$H_R(x) \geq H_L(x)$$

for all $x \in [0, 1]$. We shall work with this characterization for a uniform prior.

We now distinguish four cases:

- (A) $q_\pi(b | \alpha, \hat{\beta}) \leq q_\pi(b | \alpha)$.
- (B) $q_\pi(b | \alpha) < q_\pi(b | \alpha, \hat{\beta}) \leq \pi(b)$.
- (C) $\pi(b) < q_\pi(b | \alpha, \hat{\beta}) < q_\pi(b | \beta)$.
- (D) $q_\pi(b | \beta) \leq q_\pi(b | \alpha, \hat{\beta})$.

CASE A: Assume first that the observation $(\alpha, \hat{\alpha})$ has strictly positive probability. We shall demonstrate:

$$H_R(x) \geq H_L(x) \quad \forall x \in [0, 1] \Leftrightarrow q_\pi(b | \alpha, \hat{\alpha}) \geq \pi(b). \quad (5)$$

Note that the condition on the right hand side is equivalent to (C2) if $(\alpha, \hat{\alpha})$ has strictly positive probability. Let $\bar{q} \equiv q_\pi(b | \alpha, \hat{\alpha})$. To show (5), the following property is key:

$$q_\pi(\alpha, \hat{\beta})[\bar{q} - q_\pi(b | \alpha, \hat{\beta})] = q_\pi(\alpha)[\bar{q} - q_\pi(b | \alpha)]. \quad (6)$$

To see (6), we apply the fact that the expected posterior equals the prior of the posterior conditional on having observed α , i.e.

$$q_\pi(b \mid \alpha) = q_\pi(\alpha, \hat{\alpha} \mid \alpha)q_\pi(b \mid \alpha, \hat{\alpha}) + q_\pi(\alpha, \hat{\beta} \mid \alpha)q_\pi(b \mid \alpha, \hat{\beta}).$$

Where $q_\pi(\alpha, \hat{\alpha} \mid \alpha)$ and $q_\pi(\alpha, \hat{\beta} \mid \alpha)$ stand, respectively, for the probability of observing $(\tilde{\sigma}_1, \tilde{\sigma}_2) = (\alpha, \hat{\alpha})$ and $(\tilde{\sigma}_1, \tilde{\sigma}_2) = (\alpha, \hat{\beta})$ conditional on $\tilde{\sigma}_1 = \alpha$. If we use this in the right hand side of (6) and take out common factors, we obtain that (6) is equivalent to

$$\begin{aligned} & [q_\pi(\alpha, \hat{\beta}) - q_\pi(\alpha) + q_\pi(\alpha)q_\pi(\alpha, \hat{\alpha} \mid \alpha)]q_\pi(b \mid \alpha, \hat{\alpha}) \\ &= [q_\pi(\alpha, \hat{\beta}) - q_\pi(\alpha)q_\pi(\alpha, \hat{\beta} \mid \alpha)]q_\pi(b \mid \alpha, \hat{\beta}). \end{aligned}$$

Now note that the terms in each of the squared brackets sum to zero, and this establishes (6)

We now demonstrate the “only if” part of (5). Suppose to the contrary that $\bar{q} < \pi(b)$. We show that $H_R(x) < H_L(x)$ for all $x \in (\bar{q}, \pi(b))$. To compute H_R in this range, we write down F_R for $x \in [0, \pi(b))$:

$$F_R(q) = \begin{cases} 0 & \text{if } q \in [0, q_\pi(b \mid \alpha, \hat{\beta})], \\ q_\pi(\alpha, \hat{\beta}) & \text{if } q \in [q_\pi(b \mid \alpha, \hat{\beta}), \bar{q}], \\ q_\pi(\alpha, \hat{\beta}) + 1/2 \cdot q_\pi(\alpha, \hat{\alpha}) & \text{if } q \in [\bar{q}, \pi(b)). \end{cases}$$

To understand this, recall that $F_R(q)$ is the probability with which the posterior $q_\pi(b \mid \sigma_R)$ is less than q . Since, there is no mass point of $q_\pi(b \mid \sigma_R)$ to the left of $q_\pi(b \mid \alpha, \hat{\beta})$, the first line follows. As for the second line, note that due to symmetry $q_\pi(b \mid \sigma_R)$ equals $q_\pi(b \mid \alpha, \hat{\beta})$ if and only if the realization of $\tilde{\sigma}_R$ is $(\alpha, \hat{\beta})$ or $(\beta, \hat{\alpha})$. According to the definition of $\tilde{\sigma}_R$ this happens with probability $(1/2)q_\pi(\alpha, \hat{\beta}) + (1/2)q_\pi(\beta, \hat{\alpha}) = q_\pi(\alpha, \hat{\beta})$. The argument for the third line is analogous.

With a similar argument, we obtain that

$$F_L(q) = \begin{cases} 0 & \text{if } q \in [0, q_\pi(b \mid \alpha)], \\ q_\pi(\alpha) & \text{if } q \in [q_\pi(b \mid \alpha), \pi(b)). \end{cases}$$

Since F_k is a step function, H_k can be calculated as the size of rectangles under the graph of F_k . For $x \in [\bar{q}, \pi(b))$, we have

$$\begin{aligned} H_R(x) &= q_\pi(\alpha, \hat{\beta})[\bar{q} - q_\pi(b \mid \alpha, \hat{\beta})] + (q_\pi(\alpha, \hat{\beta}) + 1/2 \cdot q_\pi(\alpha, \hat{\alpha}))[x - \bar{q}] \\ H_L(x) &= q_\pi(\alpha)[x - q_\pi(b \mid \alpha)]. \end{aligned}$$

We can now show that $H_R(x) < H_L(x)$ for $x \in (\bar{q}, \pi(b))$. We do so by showing first that $H_R(\bar{q}) = H_L(\bar{q})$, and then that the slope of $H_R(x)$ is smaller than the slope of $H_L(x)$ for all $x \in (\bar{q}, \pi(b))$. The first claim follows by (6). The second claim is true since for $x \in (\bar{q}, \pi(b))$, we have

$$H'_L(x) = q_\pi(\alpha) > q_\pi(\alpha, \hat{\beta}) + (1/2)q_\pi(\alpha, \hat{\alpha}) = H'_R(x).$$

We next demonstrate the “if”-part of (5). Let $\bar{q} \geq \pi(b)$. We begin by determining F_k . We first focus on the case $\bar{q} < q_\pi(b \mid \beta, \hat{\beta})$. We explain later how the proof needs to be modified if $\bar{q} \geq q_\pi(b \mid \beta, \hat{\beta})$. Analogous arguments as above yield:

$$F_R(q) = \begin{cases} 0 & \text{if } q \in [0, q_\pi(b \mid \alpha, \hat{\beta})], \\ q_\pi(\alpha, \hat{\beta}) & \text{if } q \in [q_\pi(b \mid \alpha, \hat{\beta}), \pi(b)], \\ q_\pi(\alpha, \hat{\beta}) + 1/2 & \text{if } q \in [\pi(b), \bar{q}], \\ q_\pi(\alpha, \hat{\beta}) + 1/2 + 1/2 \cdot q_\pi(\alpha, \hat{\alpha}) & \text{if } q \in [\bar{q}, q_\pi(b \mid \beta, \hat{\beta})], \\ 1 & \text{if } q \in [q_\pi(b \mid \beta, \hat{\beta}), 1], \end{cases}$$

and

$$F_L(q) = \begin{cases} 0 & \text{if } q \in [0, q_\pi(b \mid \alpha)], \\ q_\pi(\alpha) & \text{if } q \in [q_\pi(b \mid \alpha), q_\pi(b \mid \beta)], \\ 1 & \text{if } q \in [q_\pi(b \mid \beta), 1]. \end{cases}$$

The functions H_R, F_R, H_L, F_L are displayed in Figures 7 and 8.

To show that $H_R(x) \geq H_L(x)$ for all $x \in [0, 1]$, we proceed from left to right. For $x \in [0, q_\pi(b \mid \alpha)]$, the claim is trivially true, since F_L and thus H_L are zero.

We next show the claim for $x \in [q_\pi(b \mid \alpha), \pi(b))$. Integrating over F_k yields

$$\begin{aligned} H_R(x) &= q_\pi(\alpha, \hat{\beta})[x - q_\pi(b \mid \alpha, \hat{\beta})], \\ H_L(x) &= q_\pi(\alpha)[x - q_\pi(b \mid \alpha)]. \end{aligned}$$

It is thus sufficient to show that $H_L(x) \leq q_\pi(\alpha, \hat{\beta})[x - q_\pi(b \mid \alpha, \hat{\beta})]$ in the larger interval $[q_\pi(b \mid \alpha), \bar{q}]$. We show that the inequality holds at the lower and upper bound of the interval. Since both functions are linear, the assertion then follows. Indeed, at the lower boundary $x = q_\pi(b \mid \alpha)$ the claim is

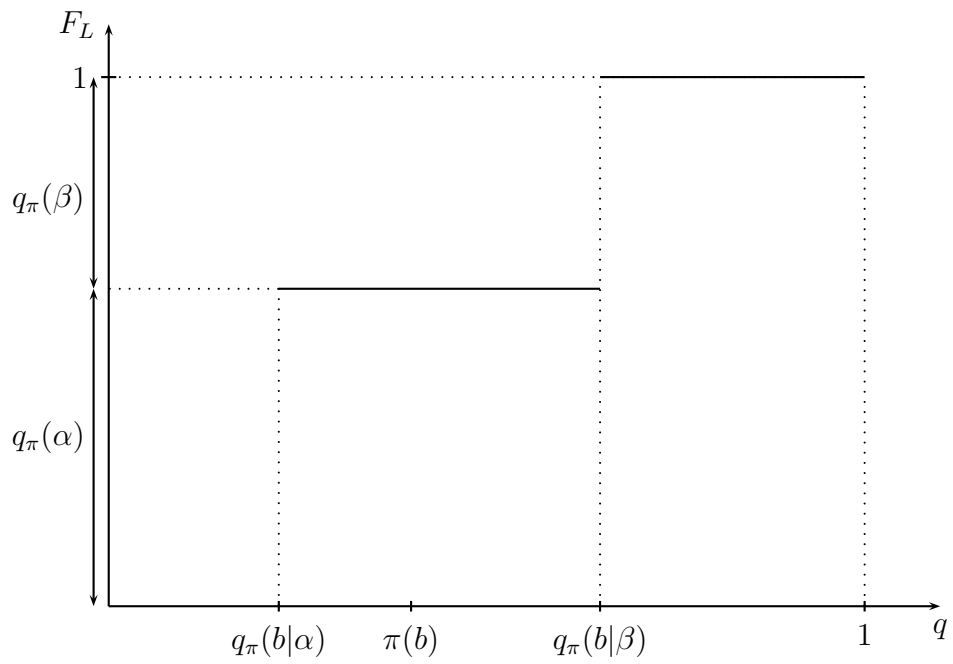
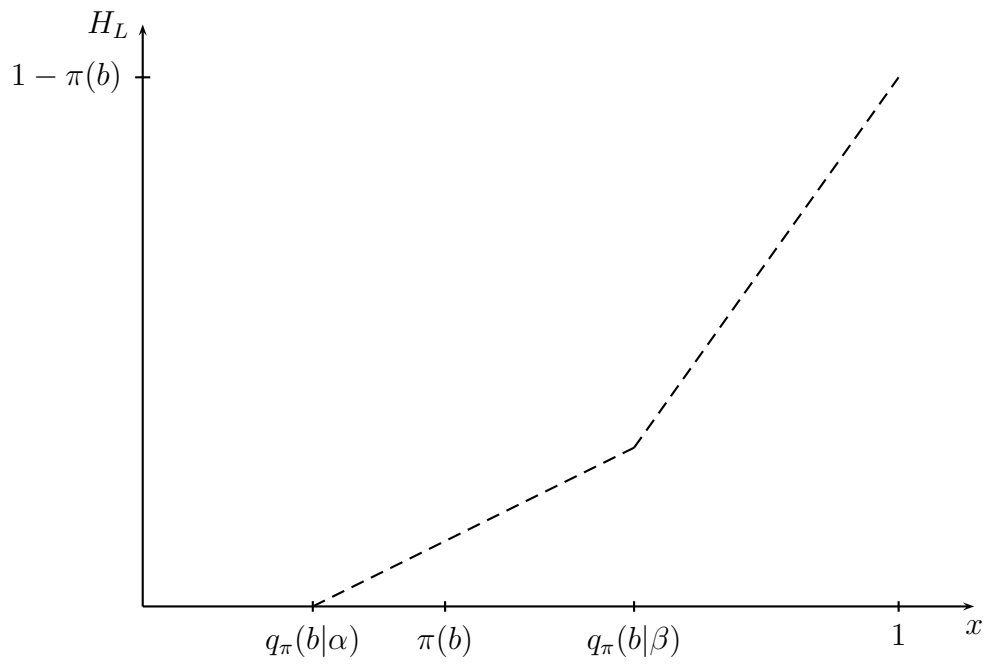


Figure 7: The functions $H_L(q)$ and $F_L(q)$

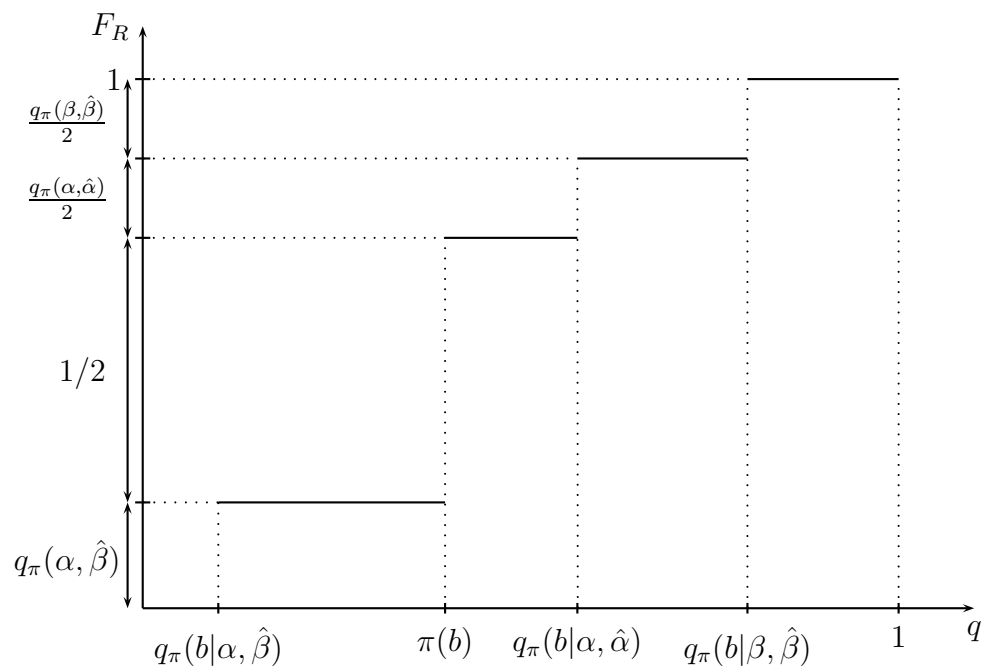
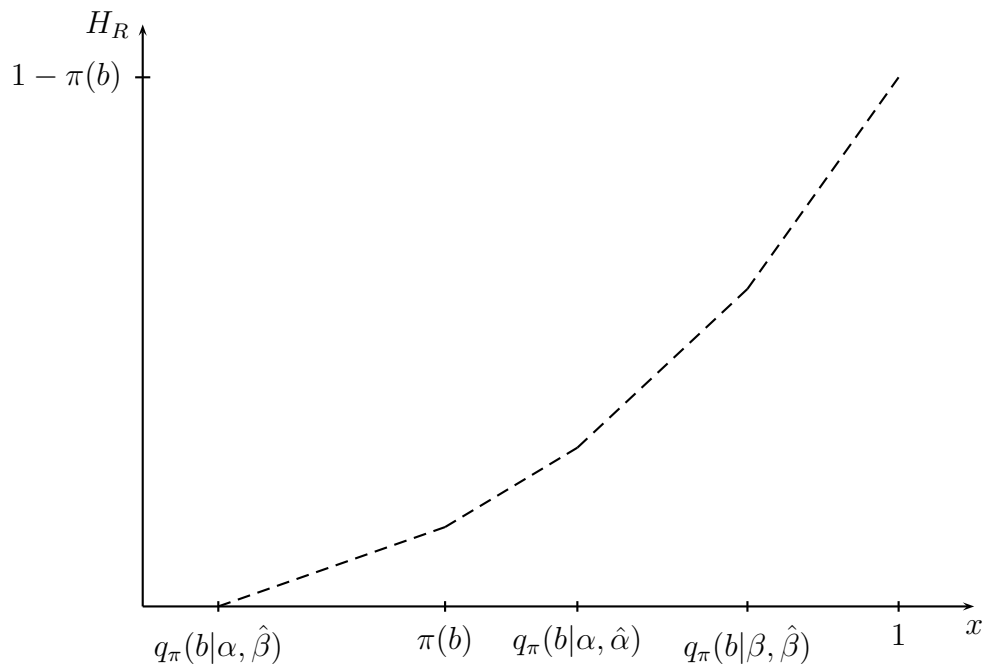


Figure 8: The functions $H_R(q)$ and $F_R(q)$

trivially true since $H_L(q_\pi(b \mid \alpha)) = 0$. At the upper boundary $x = \bar{q}$, (6) implies that $H_L(\bar{q}) = q_\pi(\alpha, \hat{\beta})[\bar{q} - q_\pi(b \mid \alpha, \hat{\beta})] = H_R(\bar{q})$.

Next, we show that $H_R(x) \geq H_L(x)$ for $x \in [\pi(b), q_\pi(b \mid \beta))$. We distinguish two cases: Let first $\bar{q} \geq q_\pi(b \mid \beta)$. Note that $H_L(x)$ is the same as in the previous paragraph, but that

$$H_R(x) = q_\pi(\alpha, \hat{\beta})[\pi(b) - q_\pi(b \mid \alpha, \hat{\beta})] + (q_\pi(\alpha, \hat{\beta}) + 1/2)[x - \pi(b)].$$

Therefore, $H_R(x)$ is certainly larger than $q_\pi(\alpha, \hat{\beta})[x - q_\pi(b \mid \alpha, \hat{\beta})]$. The same argument as in the previous paragraph can now be applied and so the claim follows. Let now $\bar{q} < q_\pi(b \mid \beta)$. Observe first that for $x < \bar{q}$, H_R takes the same form as in the case $\bar{q} \geq q_\pi(b \mid \beta)$. Thus the same argument as in the previous case implies that $H_R(x) \geq H_L(x)$ for $x \in [\pi(b), \bar{q})$. For $x \geq \bar{q}$, integrating F_R yields

$$\begin{aligned} H_R(x) &= q_\pi(\alpha, \hat{\beta})[\pi(b) - q_\pi(b \mid \alpha, \hat{\beta})] + (q_\pi(\alpha, \hat{\beta}) + 1/2)[\bar{q} - \pi(b)] \\ &\quad + (q_\pi(\alpha, \hat{\beta}) + 1/2 + 1/2 \cdot q_\pi(\alpha, \hat{\alpha}))[x - \bar{q}]. \end{aligned}$$

Since both $H_R(x)$ and $H_L(x)$ are linear functions in the interval $[\bar{q}, q_\pi(b \mid \beta))$, we certainly obtain that $H_R(x) \geq H_L(x)$ if H_R is larger than H_L at the lower boundary \bar{q} and if H_R has a larger slope than H_L . The fact that $H_R(\bar{q}) \geq H_L(\bar{q})$ is true since we have already established that $H_R(\bar{q}) \geq H_L(\bar{q})$ for all $x \in [0, \bar{q})$ and since H_k is continuous. To see that H_R has a larger slope than H_L , observe that for $x \in (\bar{q}, q_\pi(b \mid \beta))$ it holds

$$\begin{aligned} H'_L(x) = q_\pi(\alpha) &= q_\pi(\alpha, \hat{\beta}) + 1/2 \cdot q_\pi(\alpha, \hat{\alpha}) + 1/2 \cdot q_\pi(\alpha, \hat{\alpha}) \\ &\leq q_\pi(\alpha, \hat{\beta}) + 1/2 + 1/2 \cdot q_\pi(\alpha, \hat{\alpha}) = H'_R(x). \end{aligned}$$

To complete the proof, we have to show that $H_R(x) \geq H_L(x)$ for all $x \in [q_\pi(b \mid \beta), 1]$. We first compute H_L . Integrating over F_L gives for all $x \in [q_\pi(b \mid \beta), 1]$:

$$\begin{aligned} H_L(x) &= q_\pi(\alpha)[q_\pi(b \mid \beta) - q_\pi(b \mid \alpha)] + 1 \cdot [x - q_\pi(b \mid \beta)] \\ &= x - q_\pi(\alpha)q_\pi(b \mid \alpha) - q_\pi(\beta)q_\pi(b \mid \beta) \\ &= x - \pi(b). \end{aligned}$$

The second line results from a simple manipulation, the third line results from the fact that the posterior mean equals the prior. An analogous computation

shows that $H_R = x - \pi(b)$ for all x in the smaller interval $[q_\pi(b | \beta, \hat{\beta}), 1]$, establishing the claim for $x \in [q_\pi(b | \beta, \hat{\beta}), 1]$. The claim for the remaining range $[q_\pi(b | \beta), q_\pi(b | \beta, \hat{\beta})]$ follows now since we have already established in the previous paragraph that $H_R(q_\pi(b | \beta)) \geq H_L(q_\pi(b | \beta))$, and because both H_R and H_L are linear in the range $[q_\pi(b | \beta), q_\pi(b | \beta, \hat{\beta})]$. This completes the proof that $H_R(x) \geq H_L(x)$ for all $x \in [0, 1]$ if $\bar{q} < q_\pi(b | \beta, \hat{\beta})$.

If $\bar{q} \geq q_\pi(b | \beta, \hat{\beta})$, the previous arguments work in the same way to prove that $H_R(x) \geq H_L(x)$ for all $x \in [0, q_\pi(b | \beta)]$. For $x \in (q_\pi(b | \beta), 1]$, we can apply the steps in the previous paragraph to obtain that $H_L(x) = x - \pi(b)$ for all $x \in (q_\pi(b | \beta), 1]$, and that $H_R(x) = x - \pi(b)$ for all $x \in [\bar{q}, 1]$. Hence, H_R and H_L coincide on $[\bar{q}, 1]$. We also know already that $H_R(q_\pi(b | \beta)) \geq H_L(q_\pi(b | \beta))$. Now note that H_R is convex and H_L is linear on $(q_\pi(b | \beta), 1]$. This implies that H_R must lie above H_L , and this establishes the desired result.

To complete Case (A), it remains to consider the case in which the observation $(\alpha, \hat{\alpha})$ has zero probability in both states. Clearly, this implies (C2), and the “only-if”-part of Proposition 2 is trivially true. For the “if”-part, we can use the same arguments as above to show that $H_R(x) \geq H_L(x)$ for all $x \in [0, 1]$ with the only difference that \bar{q} is not well-defined. Yet, the previous arguments remain true if we replace \bar{q} by $q_\pi(b | \alpha)$ and keep in mind that $q_\pi(b | \alpha, \hat{\beta}) = q_\pi(b | \alpha)$.

CASE B: Note that neither (C2) nor (C3) are compatible with Case B. Thus, it is sufficient to show that in Case (B), there is an $x \in [0, 1]$ such that $H_R(x) < H_L(x)$. In particular, we prove this claim for $x = \hat{q} \equiv q_\pi(b | \alpha, \hat{\beta})$.

To compute $H_R(\hat{q})$, note that $q_\pi(b | \alpha) < q_\pi(b | \alpha, \hat{\beta})$ implies that the observation $(\alpha, \hat{\alpha})$ must have strictly positive probability in some state and $\bar{q} = q_\pi(b | \alpha, \hat{\alpha})$ is well-defined and is smaller than $q_\pi(b | \alpha)$. Moreover, $q_\pi(b | \alpha, \hat{\beta}) \leq \pi(b)$ implies that $q_\pi(b | \beta, \hat{\beta})$, if well-defined, is greater than $q_\pi(b | \alpha, \hat{\beta})$. Indeed,

$$\begin{aligned} H_R(\hat{q}) &= 1/2 \cdot q_\pi(\alpha, \hat{\alpha})[\hat{q} - \bar{q}], \\ H_L(\hat{q}) &= q_\pi(\alpha)[\hat{q} - q_\pi(b | \alpha)]. \end{aligned}$$

From (6), $q_\pi(\alpha, \hat{\beta})[\bar{q} - \hat{q}] = q_\pi(\alpha)[\bar{q} - q_\pi(b | \alpha)]$. Hence,

$$\begin{aligned}
H_L(\hat{q}) &= q_\pi(\alpha)[\hat{q} - \bar{q}] + q_\pi(\alpha)[\bar{q} - q_\pi(b | \alpha)] \\
&= q_\pi(\alpha)[\hat{q} - \bar{q}] + q_\pi(\alpha, \hat{\beta})[\bar{q} - \hat{q}] \\
&= (q_\pi(\alpha) - q_\pi(\alpha, \hat{\beta}))[\hat{q} - \bar{q}] \\
&= q_\pi(\alpha, \hat{\alpha})[\hat{q} - \bar{q}] \\
&> H_R(\hat{q}).
\end{aligned}$$

This establishes the claim for Case (B).

For the analysis of Case (C) and Case (D), let \hat{F}_k be the cumulative distribution of $q_\pi(a | \tilde{\sigma}_k)$, for $k \in \{L, R\}$, and define $\hat{H}_k(x) = \int_0^x \hat{F}_k(q) dq$. Note that $\hat{H}_R(x) \geq \hat{H}_L(x)$ for all $x \in [0, 1]$ if and only if $H_R(x) \geq H_L(x)$ for all $x \in [0, 1]$. Thus, Case (C) and Case (D) are analogous to Cases (B) and (A), respectively. We interchange the role of α and β (and so $\hat{\alpha}$ and $\hat{\beta}$), and instead of working with the posterior belief that the state is b , we work with the posterior belief that the state is a , and instead of $H_R(x)$, $H_L(x)$, and condition (C2), we use $\hat{H}_R(x)$ and $\hat{H}_L(x)$ and condition (C3), respectively.