Answer Key #1 Phil 414
JL Shaheen Fall 2010

1. **1.42(a)** $\mathcal{B}$ is equivalent to $\neg\neg\mathcal{B}$, and so also to $\neg\mathcal{C}$, where $\mathcal{C}$ is a DNF formula equivalent to $\neg\mathcal{B}$. (By Prop 1.5, there is such a $\mathcal{C}$.) Negated DNF meets de Morgan's Laws = $\neg\mathcal{C}$ is equivalent to some CNF formula. (You should probably have actually given an argument to show that a negated DNF formula is a CNF formula. An inductive argument based on the number of disjuncts in the DNF works. To prove the inductive step, truth tables!) So $\mathcal{B}$ is too.

   1.42(c)(ii) Let $\mathcal{C}$ be the DNF formula where each disjunct is the conjunction of literals corresponding to each row in the truth table for $\mathcal{B}$ that makes $\mathcal{B}$ true. (That is, for each such row, there is exactly one disjunct in $\mathcal{C}$ that is the conjunction of the statement letters true in that row together with the negations of the statement letters false in that row.) Since $\mathcal{C}$ is a disjunction, a row of its truth table makes $\mathcal{C}$ false just in case none of the disjuncts in $\mathcal{C}$ is true in that row. But any such row also makes $\mathcal{B}$ false by construction of $\mathcal{C}$. Similarly, a row of $\mathcal{C}$'s truth table makes it true just in case one of its disjuncts is true in that row, and any such row also makes $\mathcal{B}$ true by construction of $\mathcal{C}$. So $\mathcal{B}$ and $\mathcal{C}$ are equivalent.

   Now, if this $\mathcal{C}$ has exactly n (distinct) letters, then the truth table for $\mathcal{B}$ has 2^n rows. Since $\mathcal{C}$ has one disjunct for each row of $\mathcal{B}$'s truth table that makes $\mathcal{B}$ true, if $\mathcal{C}$ has 2^n disjuncts, $\mathcal{B}$ is true in every row of its truth table. That is, it's a tautology. The other direction follows more or less directly by construction of $\mathcal{C}$. (No further proof is really necessary, but presumably it will be clear if you understand why this is true from how you say it.)

2. **1.43(a)** S'pose $\mathcal{B}$ is satisfiable. Then there is some way to assign T's and F's to its statement letters that makes it true. But then this same way of assigning T's and F's to its statement letters makes $\neg\mathcal{B}$ false. So $\neg\mathcal{B}$ ain't no tautology.

   Now s'pose $\neg\mathcal{B}$ fails of tautologousness.[1] Then there's some way

[1] In at least one ancient manuscript, tautologology.

to assign T's and F's to its statement letters that makes it false. But that way makes $\mathcal{B}$ true, so it's satisfiable.

1.43(c) S'pose some truth assignment satisfies $\mathcal{D}$. Then it makes some disjunct of $\mathcal{D}$, say L_i, true. Any extension of the assignment will then make the conjunct of $\mathcal{E}$ containing L_i. So we need to extend the assignment to cover the C_j such that every other conjunct is true as well. Note first that if $i = 1$ or $i = 2$, then making all the C_j false makes $\mathcal{E}$ true, so henceforth we'll assume $i \geq 3$, i.e., that at least two literals containing some C_j, one negated and one just a statement letter, appear as the other two disjuncts in L_i's conjunct in $\mathcal{D}$. Now we extend the original truth assignment by assigning true to all C_j where $1 \leq j \leq i - 2$ and false to all C_j for $i - 1 \leq j \leq n - 2$, which results in an extension of the original truth assignment that satisfies $\mathcal{E}$. (To prove the extended assignment described in this way indeed satisfies $\mathcal{E}$, induction on $n \geq 4$ works.)

Now s'pose some truth assignment satisfies $\mathcal{E}$. To show that it extends a truth assignment that satisfies $\mathcal{D}$, it suffices to show that it extends a truth assignment that makes some literal of $\mathcal{D}$ true. Proof by contradiction: suppose it makes all literals of $\mathcal{D}$ false. Then it must assign C_1 to true. And if it makes C_j true, it better make $C_j + 1$ true, since none of the L_i are true but each conjunct has to be true. In particular, it makes C_{n-2} true. But then the final conjunct, $\neg C_{n-2} \vee L_n \vee \neg C_1$, is false, contradicting the supposition that the truth assignment satisfies $\mathcal{E}$. So indeed it must extend a truth assignment that satisfies $\mathcal{D}$.

1.43(d) The important properties of $\mathcal{E}$ are that any truth assignment satisfying $\mathcal{D}$, here $L_1 \vee L_2 \vee L_3$, can be extended to satisfy $\mathcal{E}$, and that any truth assignment satisfying $\mathcal{E}$ itself extends a truth assignment satisfying $\mathcal{D}$. The particular kind of $\mathcal{E}$ that we'll consider is a CNF formula such that every conjunct has at most two literals. To prove this by contradiction, s'pose there is such an $\mathcal{E}$. $\mathcal{E}$ must contain each of L_1, L_2, and L_3, or else $\mathcal{E}$ would be satisfiable by a truth assignment that does not extend a truth assignment satisfying $\mathcal{D}$ just because it assigns no truth value to the missing statement letter. So each of the statement letters of $\mathcal{D}$ appears in $\mathcal{E}$. Now, any conjunct containing some L_i contains at most either another L_j or some literal not appearing in $\mathcal{D}$, say

C_k. If some such conjunct contains no other statement letter, then the truth assignment that makes the sole statement in this conjunct false, but assigns T to at least one of the other L_j is a truth assignment that makes $\mathcal{D}$ true but cannot be extended to an assignment that truthifies the CNF formula $\mathcal{E}$. Further, if one of them, suppose without loss of generality one that contains L_1, contains some L_j, suppose without loss of generality L_3, then $\mathcal{E}$ contains a conjunct, call it $\mathcal{C}$, which contains only the statement letters L_1 and L_3, in which case there is a truth assignment that makes $\mathcal{D}$ true by making L_2 true but these literals false, which cannot be extended to an assignment that satisfies $\mathcal{E}$ just because it makes both literals in $\mathcal{C}$ false, and thus the disjunction $\mathcal{C}$ and thereby the CNF form $\mathcal{E}$ false. So each conjunct in $\mathcal{E}$ containing any L_i contains as a second literal some C_k not appearing in $\mathcal{D}$. But then $\mathcal{E}$ can be satisfied by assigning the right values to the C_k, and F to each of L_1, L_2, and L_3. So the conjuncts of $\mathcal{E}$ containing any L_i must contain some other statement letter, but cannot contain any statement letter from $\mathcal{D}$, and cannot each have as their other statement letters only those not in $\mathcal{D}$. But this just means that there can be no such $\mathcal{E}$.

3. **1.44(a)** (i) $(A \vee \neg B) \wedge B \wedge A$

(ii) $(A \vee B \vee C) \wedge (A \vee \neg B \vee C) \wedge (A \vee C)$

(iii) $(A \vee C) \wedge (\neg A \vee B) \wedge (A \vee \neg C) \wedge (\neg A \vee \neg B) \wedge (B \vee C) \wedge A \wedge (C \vee \neg B) \wedge (B \vee \neg C) \wedge \neg A \wedge (\neg C \vee \neg B)$

1.44(b) S'pose $\mathcal{B}$ holds. Then each conjunct in $Res(\mathcal{B})$ that is also in $\mathcal{B}$ holds. Now to deal with the conjuncts in $Res(\mathcal{B})$ not also in $\mathcal{B}$. By construction of $Res(\mathcal{B})$, any such conjunct, e.g., $\mathcal{C}$, is formed from two conjuncts from $\mathcal{B}$ that we are supposing to be true, one of which is a disjunction including a statement letter as disjunct, and the other of which is a disjunction including the negation of that statement letter as disjunct. Clearly one of these disjunctive conjuncts is true based on the literal containing that statement letter. The corresponding literal in the other of the two disjunctive conjuncts will then be false, so the conjunct itself can only be true on the basis of some other of its disjuncts. And that disjunct appears again in C, the conjunct which appears in $Res(\mathcal{B})$ but not in $\mathcal{B}$. But this means that C also must be true. Since C is an arbitrary conjunct of $Res(\mathcal{B})$ not in $\mathcal{B}$, and since we have already taken care of the conjuncts that appear in both

$Res(\mathcal{B})$ and $\mathcal{B}$ itself, it follows that $Res(\mathcal{B})$ is itself true.

1.44(c) S'pose $r_C(\mathcal{B})$ is satisfiable, and consider some truth assignment that witnesses this. According to this truth assignment, all conjuncts that $r_C(\mathcal{B})$ and $\mathcal{B}$ have in common are true, since $r_C(\mathcal{B})$ is itself true according to the assignment. By construction of $r_C(\mathcal{B})$, the only conjuncts in $\mathcal{B}$ that are not in $r_C(\mathcal{B})$ are conjunctions that yielded conjuncts in $r_C(\mathcal{B})$ by resolution on C. But these conjuncts in $r_C(\mathcal{B})$ are themselves true on the assignment, and any disjunction reached by resolution on C implies both of the disjunctions that yielded it by resolution on C, because whatever disjunct witnesses the truth of such a disjunction witnesses the truth of both of its parents.

1.44(d) Take the book's hint to do induction on the number of distinct statement letters that occur in some unsatisfiable CNF formula $\mathcal{B}$.

To be brief, here's just the inductive step: we assume that, for any CNF formula $\mathcal{B}'$ containing n distinct statement letters, $\mathcal{B}'$ is unsatisfiable only if $Res(\mathcal{B}')$ is a blatant contradiction, and consider a CNF formula, call it $\mathcal{B}$, containing $n+1$ statement letters, and s'pose it's unsatisfiable. Pick a statement letter occurring in $\mathcal{B}$. We'll assume WLOG that it is C. Now consider $r_C(\mathcal{B})$. Since $\mathcal{B}$ is stipulated to be unsatisfiable, so is $r_C(\mathcal{B})$ (from part (c) by contraposition). Now, $r_C(\mathcal{B})$ is a CNF formula with n statement letters, so $Res(r_C(\mathcal{B}))$ is a blatant contradiction, by the inductive hypothesis. Finally, note that all of the conjuncts of $Res(r_C(\mathcal{B}))$ also appear in $Res(\mathcal{B})$, by construction of $r_C(\mathcal{B})$ and $Res(\mathcal{B})$ (and, if you like, $Res(r_C(\mathcal{B}))$ too), so $Res(\mathcal{B})$ must be a blatant contradiction as well.

1.44(e) The left-to-right direction is just part (d). For the right-to-left direction, s'pose $Res(\mathcal{B})$ is a blatant contradiction. If the conjuncts in $Res(\mathcal{B})$ that witness its being a blatant contradiction are also in $\mathcal{B}$, then $\mathcal{B}$ is itself a blatant contradiction, and therefore obviously unsatisfiable. If those conjuncts, which we can assume WLOG to be C and $\neg C$ are not in $\mathcal{B}$, then they were each formed as the result of resolution on some other statement letter, say D and E respectively. That is, $\mathcal{B}$ must contain as conjuncts $(C \vee D)$ and $(C \vee \neg D)$, as well as $(\neg C \vee E)$ and $(\neg C \vee \neg E)$. But the conjunction of these four disjunctions is unsatisfiable (give some argument!), so $\mathcal{B}$, which entails that conjunction, must be as well.

4. **1.45(a)** S'pose $\mathcal{B} \Rightarrow \mathcal{D}$ is a tautology, that $\mathcal{B}$ and $\mathcal{D}$ has no statement letters in common, and that $\mathcal{B}$ is not contradictory. Then there is some truth assignment to the statement letters in $\mathcal{B}$ that makes it true. If $\mathcal{D}$ is not a tautology, there is also some truth assignment to the statement letters in $\mathcal{D}$ that makes it false. Since $\mathcal{B}$ and $\mathcal{D}$ share no sentence letters in common, these truth assignments can be unified in a truth assignment that is an extension of each, according to which $\mathcal{B}$ is true and $\mathcal{D}$ is false. But $\mathcal{B} \Rightarrow \mathcal{D}$ is a tautology, so nope.

   1.45(b) Let $\mathcal{C}$ be the disjunction of all eligible conjunctions, as defined in the hint on p. 405, such that each eligible conjunction contains all and only statement letters that occur in both $\mathcal{B}$ and $\mathcal{D}$. Then $\mathcal{B} \Rightarrow \mathcal{C}$ is a tautology. (You should probably say more than this here, but the meat of the problem is to show that $\mathcal{C} \Rightarrow \mathcal{D}$ is a tautology.) Now s'pose for dolphins[2] of contradiction that there is some truth assignment that makes $\mathcal{C}$ truthy but $\mathcal{D}$ falsey. Extend this to a truth assignment that covers the statement letters in $\mathcal{B}$ that are not in $\mathcal{D}$ (and so not in $\mathcal{C}$). Some such truth assignment still makes $\mathcal{C}$ truthy and $\mathcal{D}$ feel inadequate, since it doesn't change anything about the statement letters in either formula. But it also truthifies $\mathcal{B}$, per impossible since $\mathcal{B} \Rightarrow \mathcal{D}$ is a tautology. So there ain't no such truth assignment. So $\mathcal{C} \Rightarrow \mathcal{D}$ is also a tautology.

   Boogie.

[2] "porpoises"