David Hilbert and Paul du Bois-Reymond: Limits and Ideals

D.C. McCarty

1 Hilbert's Program and Brouwer's Intuitionism

Hilbert's Program was not born, nor did it live, nor is it fully understood as a defense against Brouwerian intuitionism. It was born the offspring of "Hilbert's Project," his decades-long engagement with and advocacy for the epistemology, proper pedagogy and social significance of the mathematical sciences. The goals of the Project shaped the philosophical writings of Hilbert's maturity, among them his 1919-20 lectures, **Nature and Mathematical Knowledge**:

> [I]t is the most pressing task for philosophy to investigate the questions how knowledge comes to be, what it is and what it takes as its goal. I will handle my material in the light of these questions. My lecture should, therefore, form a kind of preparation for an epistemology [*Erkenntnistheorie*]. [Hilbert 1922, 3]

Intellectual ingredients for the later Program came from the Project, among them Hilbert's axiom of solvability, that every well-posed mathematical problem admits a resolution, either a definite 'Yes,' a definite 'No,' or a convincing proof that no solution will be forthcoming. Hilbert believed that the proof theory would confirm the axiom, but his statement of the axiom itself antedates the proof theory, having featured in the Problems Address of 1900, delivered seven years before Brouwer completed his PhD dissertation and eight years before Brouwer was to launch his first assault on the validity of logical laws. Prior to World War I, the Program was no more than a twinkle in Hilbert's eye, while the Project could hardly have been a reply to Brouwer, Weyl and company, for the intuitionists then counted as no large threat to mathematical peace in Göttingen.

Questions of date notwithstanding, one cannot always make good sense of Hilbert's ideas as reasonable anti-intuitionistic countermeasures, as contributory to prima facie effective arguments against Brouwer's position. Brouwer could not have accepted Hilbert's leading premises. For instance, the metatheorems for which Hilbert labored so strenuously, the proofs that various formal systems representing higher mathematics are consistent and complete, were to be obtained via strictly finitistic reasoning. This required *inter alia* that those theorems and their proofs be couched in a language carrying a particular interpretation that Hilbert took some pains in his writings to expound. On that interpretation, certain sentences with unbounded quantifiers were to count as incomplete communications: either a manner of mathematical shorthand for other finitistically complete communications or procedures for listing finitistically complete communications. Hilbert insisted that, in a language so interpreted,"[W]e cannot write down number-signs or introduce abbreviations for infinitely many numbers." [Hilbert 1922, 1123] Further, finitistically admissible denoting terms were to refer exclusively to the perceived shapes of intuited finite sequences: "[T]he objects of [finitistic] number theory are for me - in direct contrast to Dedekind and Frege - the signs themselves, whose shapes can be generally and certainly recognized by us." [Hilbert 1922, 1121]

Brouwer could not in consistency have embraced these finitistic restrictions. This had less to do with the logic at work in the conjectured proofs than with the meanings of the statements comprising them. For Brouwer, there were in pure mathematics no such meanings for statements to have. Unbounded universal quantifications in intuitionistic arithmetic were perfectly complete communications; in general, they served to assert the existence of abstract mathematical operations with infinite domains and ranges, and, so, refused finististic paraphrase. Moreover, there was no prohibition in intuitionism against writing down "number-signs or abbreviations for infinitely many numbers" so long as the infinities in question were subject to intuitionistic treatment. Third, by Brouwer's lights, "The first act of intuitionism completely separates mathematics from mathematical language, in particular from the phenomena of language which are described by theoretical logic, and recognizes that intuitionist mathematics is an essentially languageless activity of the mind." [Brouwer 1952, 1200] Accordingly, Brouwer dubbed mathematics licensed by the first act "separable mathematics," mathematics pursued in total separation from language. So, his intuitionistic mathematics could not rely for its meaning on forms or shapes of physically realizable sign sequences, phenomena of language. Lastly, in Brouwer's mathematical universe, there was no place for the concrete signs whose forms afforded ultimate objects to Hilbert's metamathematics, because the intuitionistic universe of pure mathematics was a universe entirely of mental constructions, and hence abstract to that extent.

Hilbert realized that the majority of statements made by everyday mathematicians working in, say, analysis feature unbounded universal and existential quantifiers intended to range over uncountable domains. On his foundational design, these were not to count either as finitistic or as denotationally meaningful. Such meaning as these statements would bear was to be all inference and no reference. In general, Hilbert considered the infinitary statements ideals, mere formulae adjoined to a finitistic, contentful theory to improve its deductive efficacy. Hilbert drew an analogy between ideal statements that extend contentful mathematical language and ideal points and lines that projective geometers add to extend the Euclidean plane.

By contrast, an important pressure point for the intuitionistic critique of conventional mathematics would vanish on the assumption that infinitary statements such as the classical Bolzano-Weierstrass Theorem (that every infinite, bounded set of real numbers has an accumulation point) which Hilbert would have treated as ideal, put forward no substantial claim. For Brouwer, the difficulty haunting these theorems was that the statements they make could not be backed up by reliable forms of inference. Further, at least in its higher-order reaches, intuitionistic mathematics was intended to contradict, not just formally but also contentually, hallowed results of classical mathematics. Intuitionists of Brouwerian stripe understood the proposition "every total function from the real numbers into the real numbers is continuous" to be a theorem of intuitionistic real analysis contradicting, and striking at the root of, classical analysis by, e.g., rejecting the existence of noncontinuous step functions fully defined over the reals.

Third, in his original conception of the proof theory, Hilbert assumed that proofs would be finitary objects of the same general sort as intuited strings or sequences. A fortiori, every proof would have to terminate in a finite number of steps. To quote him, "In our present investigation, proof itself is something concrete and displayable." [Hilbert 1922, 1127] Hilbert's hope was to prove consistency for formal theories by reducing to evident absurdity the assumption that the theories in question contained finite proofs of contradictions, proofs that could be fully laid out for examination. Again, Brouwer would have had little of this. In his efforts to prove the correctness of Bar Induction [Brouwer 1927], induction upwards within certain well-founded trees from sets of nodes spanning the tree, Brouwer proposed an analysis of proofs on which intuitionistically acceptable proofs are generally infinitistic and, hence, impossible to display and survey exhaustively. Ergo, any metatheorem Hilbert proved showing that no finite proof of a contradiction exists would have been insufficient, had it been available and were Brouwer able to countenance the meaning Hilbert assigned to it, to convince Brouwer that an informal mathematical theory contains no hidden contradiction. For Brouwer would have allowed that a contradiction in a theory might only come to light at the end of an infinite proof from the axioms.

Next, Hilbertian proof theoretic efforts were to be applied, not to theories in our mathematical *Umgangsprache*, but to their counterparts formalized in artificial languages. Even an intuitionistically acceptable proof that a formal theory in which results of traditional mathematics can be rendered and derived is minimally coherent would neither comfort nor discomfit a proper Brouwerian worried over mathematical coherence. The intuitionists held that mathematical language, artificial or otherwise, was no appropriate guide to mathematical thought and no real substitute for it. A strictly Brouwerian intuitionist might have offered (but did not, so far as I know) the following analogy. We can allow that a particular painting of a scene is, as a painting, visually coherent. We can see, take in and understand it. From this it does not follow that what the painting depicts ever took place, or, if it did take place, did so in just the manner there portrayed. In these respects, relations between an historical event, say Wolfe's death at Quebec, and a painting of it are like those between mathematical language and mathematical thought as conceived by Brouwer. Just as Wolfe's demise occurred before the canvas ever stood on Benjamin West's easel, intuitionistic mathematical language comes along only after the thought, to record it, perhaps inaccurately, for posterity. Language played no role at all in what really mattered: the unfolding of mathematical thought. Mathematical thinking was not to be correctly or even approximately understood as a manipulation of linguistic representations according to fixed rules, as the Hilbert Program seemed to require. The intuitionist's grasp of mathematical objects was to be immediate; in thinking, the intuitionist engaged the objects and not the signs for them. Therefore, proven coherence among expressions in an artificial language afforded little guarantee to the intuitionist that thoughts rendered in that language were mathematically cogent.

Worth special mention at this point is an issue on which Hilbertians and Brouwerians would have agreed, an issue which much exercised Hilbert: the autonomy of mathematics. With the Program and within the circumambient Project, Hilbert and his followers hoped to insure that mathematics would be acknowledged to have resolved by strictly mathematical means all serious foundational problems pertaining to it. The truth of its basic claims as well as the coherence and efficacy of its means of proof were to be guaranteed in such a way that leading metaphysical and epistemological questions about mathematics were satisfactorily answered solely in mathematical terms. No other discipline, be it metaphysics, be it psychology, was to get a telling foundational word in, as far as pure mathematics was concerned. On the issue of mathematical autonomy, the intuitionists stood in seeming agreement with Hilbert, as is apparent from [Brouwer 1952]. From the intuitionistic viewpoint, answers to all basic foundational questions regarding mathematics were attainable via mathematical insight, by which means we were to know that all mathematical objects are mental constructions and all mathematical proofs are mathematically constructive activities.

But if Hilbert's Program is not well understood as an effective reply to intuitionism, how are we to construe it? Were one to imagine a position on the foundations of mathematics to which Hilbert's Program stood in more perfect opposition, a position friendly to the assumptions of the Program but unfriendly to its conclusions, that imaginary position would require (1) that mathematical thought is a manipulation of representations at least partially rule-governed and not, as the intuitionists held, an engagement with mathematical objects themselves; (2) that proofs are finite in length; (3) that the mathematical intellect is divisible into infinitary and finitary components, the latter maintaining an epistemic priority over the former; (4) that mathematical claims which refuse finitistic reconstrual do not denote and are viewed as limits or ideals adjoined to the finitistic component to round it out; (5) that the axiom of solvability for well-posed mathematical problems fails; (6) that mathematics is not autonomous but survives only under the foundational aegis of some nonmathematical science; and (7) that modern higher mathematics, taken as a whole, is inconsistent.

2 Paul du Bois-Reymond: Mathematician and Philosopher

At the close of the 19th Century, Paul du Bois-Reymond was counted among the most successful and influential of European mathematicians. In the index of E.W. Hobson's classic textbook, **The Theory of Functions of a Real Variable and the Theory of Fourier's Series** [Hobson 1907], no mathematician received more page references than du Bois-Reymond. The index listed Cantor, Dedekind, Dini, Hardy, Lebesque and Weierstrass; only Cantor approached du Bois-Reymond in number of citations. History books credit du Bois-Reymond with introducing the terms 'extremum' and 'integral equation' into mathematics. He used the word 'metamathematics' in his 1890 monograph **On the Foundations of Knowledge in the Exact Sciences** [du Bois-Reymond 1966] in roughly its contemporary meaning. He devised a notation for rates of growth of real-valued functions that is a direct ancestor to the "big O" notation of contemporary computer science. Under the title 'limits of indefiniteness,' he defined the lim inf and lim sup of infinite series and proved basic results governing them. (Cauchy had introduced these notions but du Bois-Reymond may have been the first to recognize their full import.) A major theorem bearing his name states that, if the sum of an everywhere-convergent trigonometric series is Riemann integrable, then the series is a Fourier series. In 1868, he formulated and proved the Second Mean Value Theorem for definite integrals, a result to which his name is now attached but which Dini once misascribed to Weierstrass. In 1873, he constructed a continuous function with divergent Fourier series at every point of a dense set, thus refuting conjectures of Dirichlet and Riemann. In 1875, he described dense sets under the title 'pantachisch,' from the Greek for 'everywhere.' He later claimed, against Georg Cantor, priority in their discovery. In his textbook, Hobson awarded the laurels to du Bois-Reymond. Although Cantor presented his first diagonal proof to the public in *On an elementary question of set theory* [Cantor 1891], Paul du Bois-Reymond had been there well before him, having published a plainly diagonal argument in an 1875 article on approximation by infinitesimals. [P. du Bois-Reymond 1875] Arguably his greatest invention was the *Infinitärcalcül* or infinitary calculus, an original, nonCantorean account of infinite and infinitesimal sizes as first-class entities that represent not the extents of collections but the rates of growth of real-valued functions.

David Paul Gustav du Bois-Reymond was born in Berlin on 2 December 1831. He began his academic career in medicine and physiology at Zürich; his elder brother Emil was a famous physiologist. While in Zürich, Paul collaborated on an important study of the blindspot of the eye. Later, he turned to mathematical physics and pure mathematics, first tackling problems of partial differential equations. Du Bois-Reymond held professorial appointments at Heidelberg, Freiburg, Berlin and Tübingen, where he was successor to Hermann Hankel. Hilbert was certainly familiar with du Bois- Reymond's mathematical research; we know that the young Hilbert visited him at least once in Berlin.

Hilbert was also in touch with du Bois-Reymond's unique philosophy of mathematics, for du Bois-Reymond was widely recognized as a leading critic of efforts to arithmetize analysis, as Alfred Pringsheim's article *Du Bois-Reymond's battle against arithmetical theories* in the **Encyclopedia of Mathematical Sciences** [Pringsheim 1898 - 1904] confirms. Du Bois-Reymond died in Freiburg on 7 April 1889, having succumbed to kidney disease while on a train trip.

Emil and Paul du Bois-Reymond played major roles in the *Ignorabimusstreit*, a spirited public debate over skepticism in the natural sciences. Emil's 1872 address to the Organization of German Scientists and Doctors, *On the limits of our knowledge of nature* [E. du Bois- Reymond 1886], both sparked the debate and baptized it, for the address closed with the dramatic pronouncement, "In the face of the puzzle over the nature of matter and force and how they should be conceived, the scientist must, once and for all, resign himself to the far more difficult, renunciatory doctrine, '*Ignorabimus*' [we shall never know]." [E. du Bois-Reymond 1886, 130] Emil argued that natural science is inherently incomplete in that there are pressing foundational questions concerning fundamental physical phenomena to which science will never find adequate answers. The address unleashed a whirlwind of argument and counterargument in the press and learned journals over *Ignorabimus* that continued well into the 20th Century. As late as 1930, Richard von Mises attacked Emil du Bois-Reymond's lecture on behalf of logical positivism, denouncing Emil's skeptical tropes as *Scheinprobleme*. [Webb 119] Phenomenologist Edith Stein employed the word '*Ignorabimus*' in her summer lectures of 1932 as a general term for psychological questions that empirical science would prove unable to answer. [Stein 1987, 167] This was the *Ignorabimus* against which Hilbert so often railed. His denunciation of it loomed large in the Problems Address [Browder 1976, 7] as well as in his final public statement, the Köningsberg talk of 1930. [Hilbert 1935, 378 - 387] The latter concluded with a direct reference to Emil's lecture: "[I]n general, unsolvable problems don't exist. Instead of the ridiculous Ignorabimus, our solution is, by contrast, 'We must know. We will know'." [Hilbert 1935, 387] Those defiant lines, "We must know. We will know," are inscribed on Hilbert's burial monument in Göttingen.

Paul du Bois-Reymond's 1882 monograph **General Function Theory** [P. du Bois-Reymond 1882] and the posthumously published **On the Foundations of Knowledge in the Exact Sciences** [P. du Bois-Reymond 1966] were devoted to transplanting a similar skepticism into the realm of pure mathematics. In those works, he expounded a philosophy of mathematics both highly original and remarkably prescient that retains philosophical interest today and was no mere translation into mathematical terms of his brother's agnostic stance toward natural science. He showed himself a forceful critic of arithmetization and logicism and, in that respect as well as others, a direct ancestor of L.E.J. Brouwer. In **General Function Theory**, Paul du Bois-Reymond drew a clear distinction between actually infinite and potentially infinite sets and, recognizing that the existence of potential but nonactual infinities makes demands on logic, called into question the general validity of the *tertium non datur*. Paul du Bois- Reymond may also have been the first to conceive of lawless sequences that determine real numbers, Cauchy sequences the successive terms of which cannot be generated by any predetermined rule or procedure, and to attempt to demonstrate their existence. To illustrate the idea, he imagined sequences whose terms are given by throws of a die: "One can also think of the following means of generation for an infinite and lawless number: every place [in the sequence] is determined by a throw of the die. Since the assumption can surely be made that throws of the die occur throughout eternity, a conception of lawless number is thereby produced ." [P. Du Bois-Reymond 1882, 91] (Contemporary intuitionists have recourse to the very same analogy in illustrating their concept of lawless sequence. See [Troelstra and van Dalen 645].) He also believed that information about the physical world could be so encoded in sequences that, if an encoding sequence were governed by a law, a knowledge of that law would yield us predictions about the universe that would be impossible to make. Were we aware of laws for the development of those sequences, he reasoned, we would be able to answer correctly questions about the precise disposition of matter at any point in space and at any time in the past. He wrote, "If we think of matter as infinite, then a constant like the temperature of space is dependent on effects that cannot be cut off at any decimal place. Were its sequence of terms to proceed by a law of formation, then this law would contain the history and picture of all eternity and the infinity of space." [P. du Bois-Reymond 1882, 91 - 92] He concluded that, since we shall never possess comprehensive physical knowledge, we will never have laws for the encoding sequences. (The structural similarities between this argument of du Bois-Reymond and Brouwer's weak counterexamples are plain.)

A goodly part of **General Function Theory** was written in the form of a dialogue between two imaginary mathematicians, Idealist and Empirist. The Idealist championed a conception of the geometrical continuum on which its basic constituents are generally transcendent and include both infinite and infinitesimal magnitudes among them. The Empirist restricted mathematical consideration to those points and line segments and their interrelations that are immanent and available to geometrical intuition. Our current and future best efforts at the philosophies of mathematics and of mind will discern only these two distinct, mutually inconsistent, fundamental outlooks on the foundations of mathematics, and no final decision between them will ever be reached. No knockdown mathematical argument will be devised for favoring one over the other. Now or later, the choice between them is largely a matter of scientific temperament. According to du Bois-Reymond, mathematics, which is the scientific study of magnitude, can only be secured by being carried back to the touchstone of the geometrical continuum. Consequently, the literary artifice of debate between Idealist versus Empirist corresponded to a natural divide and an eternal dispute within human mathematical cognition.

Du Bois-Reymond reasoned that this intellectual dualism engenders absolute undecidability results: that there are meaningful questions of mathematics answers to which depend essentially upon the outlook adopted. The Idealist answers the question one way, the Empirist another. Since no conclusive mathematical consideration will ever decide between the two, such questions pose undecidable problems whose solutions will remain forever outside the range of our mathematical abilities. A main point of **General Function Theory** was that one, if not the premiere, such question is the existence of limits for bounded, monotonically increasing sequences. Bois-Reymond wrote,

> The solution of the riddle [e.g., that of limits], if I am correct, is that it is and will always remain a riddle. The simplest expression of the riddle appears to be a psychological one. The most extensive observation of our thought processes and their relation to perception leads us ineluctably to the conclusion that there are two distinct means of conception which share the same right to count as foundational in the exact sciences, since neither of the two produces results that are disconfirmable, at least when we restrict ourselves to pure mathematics. . . . These two methods of representation I name, in keeping with the standard nomenclature, . . . *Idealism* and *Empirism*. [P. du Bois-Reymond 1882, 2 - 3]

For Paul du Bois-Reymond, who advanced his own account of the infinitely small, the question "Does the continuum contain infinitesimals?" was especially pressing. His Idealist argued, along vaguely realist lines, that infinitesimal real quantities do exist and are required for the completeness of the real numbers. The Empirist stood in opposition, insisting that we have every reason to believe that real infinitesimals are figments of pure imagination, in no way required for a satisfactory higher mathematics. Du Bois-Reymond maintained that we will never locate a completely convincing demonstration for preferring either outlook on this issue. In this, as in the matter of limits, our mathematics is and will always remain radically incomplete.

3 Paul du Bois-Reymond and the Hilbert Program

Paul du Bois-Reymond's view of mathematics included positions one would naturally expect in a perspective to which the Hilbert Program was to be a reply. Hilbert and du Bois-Reymond would have agreed that, on a proper metaphysics of proof, proofs are seen to be arrays of representations. For du Bois-Reymond, mathematical thought is a rule-governed manipulation of representations and not, as the intuitionists maintained, a direct intellectual engagement with mathematical objects. Du Bois-Reymond thought the mind populated by representations or *Vorstellungen* at various levels of abstractness. At a level close to perception, representations are extracted from perceptions or intuitions and stand for perceived or intuited objects much as a Xerox copy can stand for its original. One might call these 'object representations.' Du Bois-Reymond allowed that there are also representations of more adventitious sorts, derived from no real perception but supplied by the mind itself. He called these *Wort-Vorstellungen*, word representations. At this level, there is no pictorial image but only a word to tag a concept. Our representations for the limits of sequences of rationals would generally be of this sort. For du Bois-Reymond, proofs were to be combinations of representations satisfying certain mathematical requirements, as the introductory section of **General Function Theory** explained, "So much is certainly clear: a proof has to connect either a representation which is already available at the start or the common content of a class of representations, known as a concept, with a new, to be grasped or proven, final representation via a connected chain of representations." [P. du Bois-Reymond 1882, 11]

Second, for du Bois-Reymond as for Hilbert, a successful proof had to be finite in length and completely surveyable: "For a proof, as with an explanation, is at bottom, and generally speaking, the production of a logically satisfying sequence of representations between one representation, which engages our concern, and such representations that do not disturb our peace." [P. du Bois-Reymond 1882, 111] Du Bois-Reymond's analyses of various arguments for the existence of limits, set out in **General Function Theory** [P. du Bois-Reymond 1882, 61ff] by the Idealist, presupposed that finiteness. For example, consider a standard argument, by repeated subdivision, for the least upper bound principle on the real numbers, that every bounded, strictly increasing sequence of real numbers approaches a limit, its least upper bound. Imagine a particular sequence of that sort and assume that its terms are bounded within the closed unit interval, 0 on the left and 1 on the right. Then, divide the bounding interval at the point 1/2. Now, either the terms of the sequence eventually exceed 1/2, or they always remain below 1/2 in value. If the former obtains, pick the closed right half-interval, that between 1/2 and 1. If the latter, select the closed half-interval on the left, that lying between 0 and 1/2 inclusive. Either way, the selected half-interval is sure to contain infinitely many terms of the original sequence. Now, iterate this procedure of divide-and-select. One obtains thereby a series of nested closed intervals, each of which contains an infinite tail of the original sequence and is half as long as the preceding interval in the series. The intervals of the series all have a point in common, their intersection. It follows that the common point is the limit of the sequence originally given: the terms of the sequence are sure to get arbitrarily close to it. Therefore, the required limit exists.

In response to such an argument, Paul du Bois-Reymond's Idealist did not point out the obvious flaw: as a reason for the existence of a limit, it is question-begging unless the reasoner has already somehow secured, by independent demonstrative means, the existence of a unique point lying in the intersection of all the nested intervals. Instead, the Idealist wished us to see that, as an attempted proof, as a connected chain of representations transforming representations for the premises into those for the conclusions in a finite number of intermediate steps, the procedure of repeated division fails. In the end, it cannot produce, from the starting object representations for an interval and a sequence, an object representation for a single dimensionless limiting point. For there can be only finitely many steps in the chain of representations that comprise the proof. But, after only finitely many steps, at most a finite number of interval divisions can be performed. At the finite stage in the division process when the purported proof has to terminate, one is left with, at best, the object representation of a finite interval of nonzero length (perhaps a visualized line segment) and not the visualized point or dot representing the unique real number that the argument's conclusion requires. Moreover, no finite series of additional halvings of the interval will succeed in transforming an interval-representing line segment into a point-representing dot. "On the basis of our assumptions, we can certainly keep reducing the length of the interval without limit. This is, however, a process which alters nothing in the nature of our representations. Large or small, the interval . . . remains always an interval between two rational points." [P. du Bois-Reymond 1882, 61] Whether defensible or not, this line of thought would make no sense unless proofs were required to consist entirely of representations and be finite in length. (I would ask the reader to compare the Idealist's objection to the present proof with the response of an intuitionist who accepts continuity principles to the same argument.)

Third, it is not wholly anachronistic to assert that, with his *Infinitärcalcül*, Paul du Bois-Reymond believed that he had constructed a nonstandard continuum. Of course, the ideas of nonstandard model and of abstract languages separate from their varied interpretations were not then available. Du Bois-Reymond held that the properties of his domain of *infinities* or *infinite orders* (not to be confused with Cantor's arithmetic of infinite cardinals, a development wholly distinct), agreed to some extent with those of the standard real numbers. Several features of the nonstandard domain, e.g., density, capturable by geometrical object representations are manifested in the standard geometrical continuum. But, importantly, the two continua do not have all analytically discernible features in common. The domain of infinite orders contains infinitesimals, which du Bois- Reymond was able convincingly to show. Using diagonalization, he also proved that the infinite orders do not satisfy the least upper bound principle.

To a first approximation, the opposition between idealism and empirism was a register of divergent attitudes toward the existence of infinitesimals. The Idealist championed the notion that a 'nonstandard' number continuum gives the true underlying structure of the continuum that is intuited geometrically. The Idealist believed that, behind the latter continuum, lies a structure fully available to the analytical intellect but only partially available to geometrical intuition. The idealistic continuum is an ideal, or series of ideals, posited by the mind and inserted into the intuited continuum to provide needed limits, to create a completion. By contrast, the Empirist was the champion of a real number system exhausted by geometrical intuition and mathematical empirism was to be a system, as du Bois-Reymond wrote, "of complete renunciation." [P. du Bois- Reymond 1882, 3] The Empirist renounced the infinitesmials of the Idealist and restricted himself to the directly intuited reals only. He denied outright that the Idealist's analytical machinery exists behind the mathematical scenes erected by geometrical insight.

Du Bois-Reymond's division of mathematical thought into empirism and idealism coincided with a division into its finitary and infinitary aspects, the first of which maintained a priority over the second. The Idealist took the concepts of infinitely large and infinitely small magnitudes to make sense and to be exemplified in reality; the empirist denied sense and exemplification to both. Du Bois-Reymond has the empirist exclaim, "[F]or the construction of mathematics, the finite suffices." [P. du Bois-Reymond 1882, 146] The continuum of the idealist was to be uncountable in magnitude but that of the empirist only potentially infinite. It is essential to remember that, in the writings of du Bois-Reymond, Idealist and Empirist were not so much representatives of different schools of thought, such as logicism or constructivism, as discriminable facets of one and the same mathematical consciousness. Every mathematician, thought Du Bois-Reymond, sometimes reasons as the Idealist and, at other times, as the Empirist, much as Hilbert could consistently demand finitism in metamathematics and infinitary, classical analysis in the object language. Further, since du Bois-Reymond believed that the continuum of the Empirist and that of the Idealist agreed in all apparent geometrical and elementary analytical properties, no object representation and no mathematical datum would ever distinguish between them by confirming one and disconfirming the other. The Idealist will never, unless he commits a mathematical error, claim anything to be a mathematical fact of an elementary character that the Empirist will not be able to accept. That is, again with some anachronism, one can allow that idealism was to be conservative over empirism when it came to the mathematics acceptable to the latter, much as Hilbert's infinitary higher arithmetic was to be conservative over its finitistic fragment. That said, du Bois- Reymond's empirism was to hold, in its strict adherence to geometrical intuition, a manner of priority over idealism. The knowledge it delivers is more certain in that appeal to geometrical object representation is either innate for humans or acquired very early in life, while the unfettered analytical thought and word representations of the Idealist are not. Paul du Bois-Reymond took the results of idealistic real analysis to be a belated adjustment to the original geometrical data: "Accordingly, we can glimpse, in the intercalation of irrational numbers among the rationals, only a retrospective adaptation of the intellective number concept to the concept of geometrical magnitude, which is either innate or acquired in infancy." [P. du Bois-Reymond 1877, 150] Moreover, since Idealist and Empirist were to agree on all the relevant data, the idealist could look to the continued mathematical success of the Empirist as support for his own infinitary investigations. [P. du Bois-Reymond 1882, 148]

Fourth, in the posthumous monograph [P. du Bois-Reymond 1966], du Bois-Reymond argued that infinitary mathematical claims are generally ideal in that their terms need not denote anything given by an object representation drawn from a perceived or intuited realm. Instead, the items represented by those terms should be viewed as symbolic limits added to the finitistic or empiristic sector in order to round it out. Apart from those real magnitudes like $\sqrt{2}$ associated with the visualizable result of a geometrical construction, real numbers are not allied with object representations but only with word representations that correspond to nothing in intuited mathematical reality: "The sequence of contentual representations of exactness has as its end result [its limit] a word for something unrepresentable.” [P. du Bois-Reymond 1966, 80] A few pages later in the same work, we find him asserting, “The concept of the infinite which turns up here is the most significant of the unrepresentable word representations, the foremost idealistic concept, because it is best attached to the question of our conception of the existence of the ideal.” [P. du Bois-Reymond 1966, 85 - 86]

In the **General Function Theory** [P. du Bois-Reymond 1882], any magnitude which is purely idealistic is nothing but a symbol: “All the mathematical magnitudes which we have introduced so far can be located in the realm of the inner- or extramental perceptual world. We wish to call such magnitudes real. There are also magnitudes created by the human thought process and which stand outside any direct relation to the perceptual world. The logical processes which achieve the combinations of symbols, one of the favorite activities of the human spirit, lead to certain symbols which are, in mathematics, also called magnitudes, which serve to assemble into a single sign a mathematical conclusion which frequently recurs.” [P. du Bois-Reymond 1882, 38] Incidentally, this quotation underscores the extent to which du Bois- Reymond considered empirism to hold priority over idealism: the Empirist limited his mathematical considerations to those magnitudes which are, as here explained, real. The Idealist did not.

Fifth, as earlier noted, Paul du Bois-Reymond explicitly refused the axiom of the solvability of all well-posed mathematical problems. His refusal was a direct corollary to his characterization of the dualism between Idealist and Empirist. The former believed in the existence of a continuum which contains, in addition to the rational and constructible magnitudes, infinite and infinitesimal magnitudes. The latter staunchly maintained that none of these purely analytical adjustments really exists. According to du Bois-Reymond, no completely convincing mathematical demonstration, no scientifically respectable argument, can be given that will prove either the truth or the falsity of the Idealist’s claims to the satisfaction of all. The truth of the statement “there are real infinitesimal magnitudes,” positive real magnitudes smaller than any ordinary, positive rational number, which is crucial to the Idealist’s worldview, cannot be decided. It will never be proved; it will never be disproved. There will be no final scientific determination of the structure of the continuum. An examination of the text of Hilbert’s Problems Address suggests that his axiom of solvability, there enunciated, might well have been intended to exorcize this form of incompleteness. [McCarty 2002]

Finally, Hilbert and Brouwer endorsed, in the strongest terms, the autonomy of mathematics, while du Bois-Reymond rejected it in terms no less vigorous. Paul du Bois-Reymond insisted that mathematics is not an autonomous science, but requires a nonmathematical foundation supplied by another discipline. He wrote, “As was to be suspected and as we soon recognize, the [foundational] difficulty of the concept of limit is not of a mathematical nature. If it were, it would have been dealt with long ago. The difficulty is really rooted in the simplest constituents of our thinking, the representations.” [P. du Bois-Reymond 1882, 2] So, the founding discipline for mathematics was to be the proper study of representations; for du Bois-Reymond, that was to be the physiological psychology prospering in Germany during the 19th Century, among whose foremost representatives had been Johannes Müller, Hermann von Helmholtz and his older brother Emil. In the first section of **General Function Theory**, that devoted to the analysis of magnitude, the authorities most often cited seem to be physiologists, among them Gustav Fechner and Müller. Paul du Bois-Reymond believed that physiology will reveal to us by experiment the nature and extent of represented magnitude and, on that experimental basis, the foundations of mathematics could be erected. Only in this way, by a scientific determination of what humans can actually perceive of magnitude, can the crucial foundational distinction between the Empirist, who restricts himself to magnitudes open to geometrical object representation, and the Idealist, who is willing to countenance unperceivable and unintuitable magnitudes, be made with any firm assurance. The largest feature on du Bois-Reymond's map of mathematical thought, the great divide between empirism and idealism, was to be drawn by the hand of physiology and not by that of mathematics.

4 Acknowledgements

I wish to express my gratitude to Christian Thiel and Peter Bernhard of the University of Erlangen, as well as Volker Peckhaus of the Universities of Paderborn and Erlangen, for allowing me to present a preliminary version of this essay to the Colloquium Logico-Philosophicum at Erlangen. I am also grateful to Bernd Buldt, Volker Halbach and their colleagues in the Logik Forschergruppe at the University of Konstanz, especially Ulf Friedrichsdorf and Max Urchs, for comments and suggestions on a version of this paper presented there. Help was generously provided by Gerhard Betsch, Mic Detlefsen, Reinhard Kahle and Marianne Moberg-Blauert. Last but not least, I thank Steve Forrest, Robert Noel and Paula Patton of the Indiana University Library in Swain Hall for their patience and determination in tracking down works by 19th Century mathematicians and philosophers.

5 Bibliography

Unless otherwise noted, translations from the German are my own.

Browder, Felix E. (Ed.) [1976] **Mathematical Developments arising from the Hilbert Problems**. Proceedings of Symposia in Pure Mathematics. Volume XXVIII. Providence, RI: American Mathematical Society. xii + 628. (I have amended the translation of Hilbert's 1900 lecture in the light of [Hilbert 1935].)

Brouwer, L. E. J. [1927] *Über Definitionsbereiche von Funktionen.* [On domains of definition for functions.] **Mathematische Annalen**. Volume 97. pp. 60 - 75.

Brouwer, L. E. J. [1928] *Intuitionistische Betrachtungen über den Formalismus.* [Intuitionistic reflections on formalism.] **Koninklijke Akademie van wetenschappen te Amsterdam. Proceedings of the section of sciences.** [The Royal Academy of Science in Amsterdam.] Volume 31. pp. 374 - 379.

Brouwer, L. E. J. [1952] *Historical background, principles and methods of intuitionism.* In William Ewald (tr. and ed.) **From Kant to Hilbert: A Source Book in the Foundations of Mathematics. Volume II**. Oxford, UK: Clarendon Press. 1996. pp. 1197 - 1207.

Cantor, Georg [1891] *Über eine elementare Frage der Mannigfaltigkeitslehre.* [On an elementary question of set theory.] **Jahresbericht der Deutschen Mathematiker-Vereinigung**. [Annual Report of the German Mathematical Association.] Erster Band. pp. 75 - 78.

Detlefsen, Michael [1986] **Hilbert's Program. An Essay on Mathematical Instrumentalism.** Dordrecht, NL: D. Reidel Publishing Company. xiv + 186.

Du Bois-Reymond, Emil [1886] *Über die Grenzen des Naturerkennens.* [On the limits of our knowledge of nature.] **Reden von Emil Du-Bois Reymond**. [Addresses of Emil du Bois-Reymond.] Erste Folge. Leipzig, DE: Verlag von Veit and Comp. viii + 550.

Du Bois-Reymond, Paul [1875] *Über asymptotische Werte, infinitäre Approximationen und infinitäre Auflösungen von Gleichungen.* [On asymptotic values, infinitary approximations and infinitary solutions of equations.] **Mathematische Annalen**. Volume 8. pp. 363 - 414.

Du Bois-Reymond, Paul [1877] *Ueber die Paradoxon des Infinitärcalcüls.* [On the paradoxes of the infinitary calculus.] **Mathematische Annalen**. Volume 10. pp. 149 - 167.

Du Bois-Reymond, Paul [1882] *Die allgemeine Funktionentheorie.* [General Function Theory.] Tübingen, DE: H. Laupp. xiv + 292.

Du Bois-Reymond, Paul [1966] **Über die Grundlagen der Erkenntnis in den exakten Wissenschaften**. [On the Foundations of Knowledge in the Exact Sciences.] Sonderausgabe. Darmstadt, DE: Wissenschaftliche Buchgesellschaft. vi + 130.

Hardy, G.H. [1954] **Orders of Infinity: The 'Infinitärcalcül' of Paul du Bois- Reymond**. Cambridge Tracts in Mathematics and Mathematical Physics. No. 12. Cambridge, UK: Cambridge University Press. 77 pp.

Hilbert, David [1922] *The new grounding of mathematics.* In William Ewald (tr. and ed.) **From Kant to Hilbert: A Source Book in the Foundations of Mathematics. Volume II**. Clarendon Press. Oxford. 1996. pp. 1115 - 1134.

Hilbert, David [1935] **David Hilbert. Gesammelte Abhandlungen. Dritter Band**. [David Hilbert. Collected Works. Third Volume.] Berlin, DE: Verlag von Julius Springer. vii+435.

Hilbert, David [1992] **Natur und mathematisches Erkennen**. [Nature and Mathematical Knowledge.] Berlin, DE: Birkhäuser Verlag. xiv +101.

Hobson, E. W. [1907] **The Theory of Functions of a Real Variable and the Theory of Fourier's Series**. Cambridge, UK: Cambridge University

Press. xv + 772.

McCarty, D. C. [2002] *Problems and Riddles: Hilbert and the du Bois- Reymonds.* **Synthese. Special Issue for GAP 2000.** To appear.

Pringsheim, Alfred [1898 - 1904] *Irrationalzahlen und Konvergenz unendlicher Prozesse.* [Irrational numbers and the convergence of infinite processes.] in W. F. Meyer (ed.) **Encyklopädie der mathematischen Wissenschaften. Erster Band in zwei Teilen. Arithmetik und Algebra.** [Encyclopedia of the Mathematical Sciences. First Volume in Two Parts. Arithmetic and Analysis.] Leipzig, DE: Druck und Verlag von B.G. Teubner. pp. 47 - 146.

Reid, Constance [1970] **Hilbert**. Berlin, DE: Springer-Verlag. xi + 290.

Sieg, Wilfried [1999] *Hilbert's Programs: 1917-1922.* **The Bulletin of Symbolic Logic**. Volume 5. Number 1. March 1999. pp. 1- 44.

Stein, Edith [1987] **Essays on Woman. The Collected Works of Edith Stein. Volume Two**. Freda Mary Oben (tr.) Washington DC: ICS Publications. ix + 290.

Troelstra, A. S. and D. van Dalen [1988] **Constructivism in Mathematics: An Introduction. Volume II.** Amsterdam, NL: North-Holland. xvii + 879 + LII.

Webb, Judson C. [1980] **Mechanism, Mentalism, and Metamathematics. An Essay on Finitism**. Dordrecht, NL: D. Reidel Publishing Company. xiii + 277.