

**Practice final Solutions**  
**Phil 303**  
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1. a) Prove (with a derivation) that  $\{P \vee Q, P \rightarrow R, R \rightarrow W, Q \rightarrow (R \wedge R)\} \vdash W$

1	$P \vee Q$	hyp		
2	$P \rightarrow R$	hyp		
3	$R \rightarrow W$	hyp		
4	$Q \rightarrow (R \wedge R)$	hyp		
5	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>P</math></td> <td>hyp</td> </tr> </table>	$P$	hyp	hyp
$P$	hyp			
6	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>R</math></td> <td>MP 2,5</td> </tr> </table>	$R$	MP 2,5	MP 2,5
$R$	MP 2,5			
7	$W$	MP 3,6		
8	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>Q</math></td> <td>hyp</td> </tr> </table>	$Q$	hyp	hyp
$Q$	hyp			
9	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"><math>R \wedge R</math></td> <td>MP 4,8</td> </tr> </table>	$R \wedge R$	MP 4,8	MP 4,8
$R \wedge R$	MP 4,8			
10	$R$	conj. elim 9		
11	$W$	MP 3,10		
12	$W$	disj. elim. 1 5-7 8-11		

b) The soundness theorem (metatheorem 1) allows you to conclude - from the fact established in a) - that  $\{P \vee Q, P \rightarrow R, R \rightarrow W, Q \rightarrow (R \wedge R)\} \Vdash W$

2. i) Give a categorical derivation of  $\neg(\neg A \vee B) \rightarrow (A \wedge \neg B)$

1	$\neg(\neg A \vee B)$	hyp
2	$\neg A$	hyp
3	$\neg A \vee B$	Disj intro 2
4	$\neg(\neg A \vee B)$	reit 1
5	$\neg\neg A$	neg intro 2-4
6	$A$	neg. elim. 5
7	$B$	hyp
8	$\neg A \vee B$	disj intro 6
9	$\neg(\neg A \vee B)$	reit 1
10	$\neg B$	conj. elim 9
11	$A \wedge \neg B$	conj intro 6, 10
12	$\neg(\neg A \vee B) \rightarrow (A \wedge \neg B)$	cond. intro 1-11

ii) Give a derivation of  $W \vee R$  from  $\{(P \rightarrow P) \rightarrow (A \wedge \neg A), Q \rightarrow W, Q \rightarrow R\}$

The trick for this one is just to ignore every premise except  $(P \rightarrow P) \rightarrow (A \wedge \neg A)$

1	$(P \rightarrow P) \rightarrow (A \wedge \neg A)$	hyp
2	$Q \rightarrow W$	hyp
3	$Q \rightarrow R$	hyp
4	$\neg(W \vee R)$	hyp
5	$P$	hyp
6	$P$	reit 5
7	$P \rightarrow P$	cond intro 5-6
8	$A \wedge \neg A$	MP 1,7
9	$A$	conj elim 8
10	$\neg A$	conj. elim 8
11	$\neg\neg(W \vee R)$	neg intro 4-10
12	$W \vee R$	neg elim 11

3. Prove the following statements, or give a counter-example:

a) If  $\Gamma \vdash P$  and  $\Delta \vdash Q$  then  $\Gamma \cup \Delta \vdash P \wedge Q$

True. Say you have a derivation of  $P$  from the assumptions in  $\Gamma$ . Add to this derivation the assumptions from  $\Delta$ , and derive  $Q$ . This longer derivation will contain a derivation of  $P$  and a derivation of  $Q$ , with undischarged assumptions  $\Gamma \cup \Delta$ . The last line will be a conjunction introduction yielding the conclusion  $P \wedge Q$ .

b) If  $\Gamma \models P$  and  $\Delta \models Q$  then  $\Gamma \cup \Delta \models P \wedge Q$

True. Say that some interpretation  $I$  simultaneously satisfies  $\Gamma \cup \Delta$ . Since it makes every sentence in  $\Gamma$  true, it makes  $P$  true. Since it makes every sentence in  $\Delta$  true, it makes  $Q$  true. Since it makes  $P$  and  $Q$  true, then by the truth-table for  $\wedge$ , it makes  $P \wedge Q$  true.

c) If  $\Gamma$  is satisfiable, then  $\Gamma \cup \{P \vee \neg P\}$  is satisfiable.

True. Say you have an interpretation  $I$  making every sentence in  $\Gamma$  true. There must be such an interpretation because  $\Gamma$  is simultaneously satisfiable.  $I$  must also satisfy  $P \vee \neg P$ , since it is a logical truth. So  $I$  satisfies  $\Gamma \cup \{P \vee \neg P\}$  and hence  $\Gamma \cup \{P \vee \neg P\}$  is satisfiable.

d) If  $\Gamma$  is satisfiable, and  $\Delta$  is satisfiable, then  $\Gamma \cup \Delta$  is satisfiable.

False. Let  $\Gamma = \{P\}$  and  $\Delta = \{\neg P\}$ .  $\Gamma \cup \Delta = \{P, \neg P\}$ , which is not satisfiable.

e) If  $\Gamma \models P$  then  $\Gamma \cup \{\neg P\}$  is satisfiable.

False. Say that  $\Gamma = \{P\}$ . Then  $\Gamma \models P$  but  $\Gamma \cup \{\neg P\} = \{P, \neg P\}$ , which is not satisfiable.

f) If  $\Gamma$  is satisfiable, then  $\{\neg S \mid S \in \Gamma\}$  is satisfiable.

False. Let  $\Gamma = \{P \vee \neg P\}$ . Then  $\Gamma$  is satisfiable, but  $\{\neg S \mid S \in \Gamma\}$  is not.

4. We can proceed by induction on the length of derivations.

Base Case: Say that  $A$  is proven by a derivation of one line. Then  $A$  must be a hypothesis, and so the derivation establishes (the trivial fact) that  $\{A\} \vdash A$ . Trivially any truth assignment making every sentence in  $\{A\}$  true makes  $A$  true, so  $\{A\} \models A$ .

Induction step: Say that for any sentence  $A$  and set of sentences  $\Gamma$ , whenever there is a derivation of length  $k$  or less, of  $A$  from hypotheses  $\Gamma$ , then  $\Gamma \vdash A$ . Say that  $\Gamma^* \vdash A^*$  via a derivation of exactly  $k + 1$  lines. If  $A^*$  is a hypothesis, then the entire proof must consist of hypotheses, since the hypotheses are all to be at the beginning of the proof, and  $A^* \in \Gamma^*$ . Hence, any truth assignment making every sentence in  $\Gamma^*$  true makes  $A^*$  true, so  $\Gamma^* \models A^*$ . If  $A^*$  is not a hypothesis, it must follow by modus

ponens from two other sentences  $B \rightarrow A^*$  and  $B$ , where  $\Gamma_1 \vdash B \rightarrow A^*$ , and  $\Gamma_2 \vdash B$ . ( $\Gamma_1 \cup \Gamma_2 = \Gamma^*$ ; the point is that you don't need to use all the sentences in  $\Gamma^*$  for each of the premises, even if you use all of  $\Gamma^*$  to derive  $A^*$ .) Since the derivations of  $B \rightarrow A^*$  and  $B$  are  $k$  lines or shorter, the induction hypothesis applies: from  $\Gamma_1 \vdash B \rightarrow A^*$  we can infer  $\Gamma_1 \Vdash B \rightarrow A^*$  and from  $\Gamma_2 \vdash B$  we can infer  $\Gamma_2 \Vdash B$ . What we want to show is  $\Gamma^* \Vdash A^*$ . Say we have a truth-assignment  $I$  making every sentence in  $\Gamma^*$  true. Since  $\Gamma_1 \subseteq \Gamma^*$ ,  $I$  makes every sentence in  $\Gamma_1$  true, and so it makes  $B \rightarrow A^*$  true. Since  $\Gamma_2 \subseteq \Gamma^*$ ,  $I$  makes every sentence in  $\Gamma_2$  true, so it makes  $B$  true. By the truth-table for  $\rightarrow$ , any truth assignment making  $B$  and  $B \rightarrow A^*$  true must also make  $A^*$  true, so  $I$  makes  $A^*$  true. Thus  $\Gamma^* \Vdash A^*$ .

5. Say that  $\Gamma$  is a V-saturated set and  $(\neg P \vee \neg Q) \in \Gamma$ . Explain how we know that either  $P \notin \Gamma$  or  $Q \notin \Gamma$ .

Say that  $P \in \Gamma$  and  $Q \in \Gamma$ . In this case we can derive  $R$  and  $\neg R$  from  $\Gamma$ , so  $\Gamma$  is inconsistent, which contradicts the V-saturatedness of  $\Gamma$ . So either  $P \notin \Gamma$  or  $Q \notin \Gamma$ .

Here is how you derive  $R$  and  $\neg R$  from  $\{P, Q, \neg P \vee \neg Q\}$ :

1	$\neg P \vee \neg Q$	hyp
2	$P$	hyp
3	$Q$	hyp
4	<div style="border-top: 1px solid black; padding-top: 2px;"><math>\neg P</math></div>	hyp
5	<div style="border-left: 1px solid black; padding-left: 10px;"> <div style="border-top: 1px solid black; padding-top: 2px;"><math>R</math></div> </div>	hyp
6	<div style="border-left: 1px solid black; padding-left: 10px;"> <div style="border-top: 1px solid black; padding-top: 2px;"><math>P</math></div> </div>	reit 2
7	<div style="border-left: 1px solid black; padding-left: 10px;"><math>\neg P</math></div>	reit 4
8	$\neg R$	neg intro 5-7
9	$\neg Q$	hyp
10	<div style="border-top: 1px solid black; padding-top: 2px;"> <div style="border-left: 1px solid black; padding-left: 10px;"><math>R</math></div> </div>	hyp
11	<div style="border-left: 1px solid black; padding-left: 10px;"> <div style="border-top: 1px solid black; padding-top: 2px;"><math>Q</math></div> </div>	reit 3
12	<div style="border-left: 1px solid black; padding-left: 10px;"><math>\neg Q</math></div>	reit 9
13	$\neg R$	neg intro 10-12
14	$\neg R$	disj. elim. 1, 4 - 8, 9 - 13

1	$\neg P \vee \neg Q$	hyp
2	$P$	hyp
3	$Q$	hyp
4	$\neg P$	hyp
5	<div style="border-left: 1px solid black; padding-left: 5px;"><math>\neg R</math></div>	hyp
6	<div style="border-left: 1px solid black; padding-left: 5px; border-bottom: 1px solid black;"><math>P</math></div>	reit 2
7	<div style="border-left: 1px solid black; padding-left: 5px;"><math>\neg P</math></div>	reit 4
8	$\neg\neg R$	neg intro 5-7
9	$\neg Q$	hyp
10	<div style="border-left: 1px solid black; padding-left: 5px;"><math>\neg R</math></div>	hyp
11	<div style="border-left: 1px solid black; padding-left: 5px; border-bottom: 1px solid black;"><math>Q</math></div>	reit 3
12	<div style="border-left: 1px solid black; padding-left: 5px;"><math>\neg Q</math></div>	reit 9
13	$\neg\neg R$	neg intro 10 -12
14	$\neg\neg R$	disj. elim. 1, 4 - 8, 9 - 13
15	$R$	neg. elim. 14