

Practice final Solutions
Phil 303
Posted April 23, 2009

1. Answer **one** of these two questions:

(a) Say that $CS =_{\text{definition}} \{X \mid X \notin S\}$.

Prove that $C(S_1 \cup S_2) = CS_1 \cap CS_2$

We prove the equation by showing containment in both directions. \Rightarrow Pick $X \in C(S_1 \cup S_2)$

By the definition of C , $X \notin (S_1 \cup S_2)$.

By the definition of \cup , $X \in (S_1 \cup S_2)$ if and only if $X \in S_1$ or $X \in S_2$

Negating both sides, we have $X \notin (S_1 \cup S_2)$ if and only if $\neg(X \in S_1 \text{ or } X \in S_2)$

By DeMorgan's laws, we get: $X \notin (S_1 \cup S_2)$ if and only if $X \notin S_1$ and $X \notin S_2$
so: $X \notin S_1$ and $X \notin S_2$.

By the definition of C , $X \in CS_1$ and $X \in CS_2$

By the definition of \cap , $X \in CS_1 \cap CS_2$

Since X was arbitrary, this proves:

$$C(S_1 \cup S_2) \subseteq CS_1 \cap CS_2$$

\Leftarrow Pick $X \in CS_1 \cap CS_2$

By the definition of \cap , $X \in CS_1$ and $X \in CS_2$

By the definition of C , $X \notin S_1$ and $X \notin S_2$

By DeMorgan's laws, $\neg(X \in S_1 \text{ or } X \in S_2)$

By the definition of \cup , $\neg(X \in S_1 \cup S_2)$

By the definition of C , $X \in C(S_1 \cup S_2)$

Since X was arbitrary, this proves:

$$C(S_1 \cup S_2) \supseteq CS_1 \cap CS_2.$$

By proving both directions, we prove the equation.

(b) Using the Pumping Lemma, explain why there cannot be an FSA over the alphabet $\{a, b\}$ that accepts only strings that consist of just an unbroken sequence of a 's, and then an unbroken sequence of exactly the same number of b 's.

Say that there were an FSA that accepted exactly the strings of the form:

$$\underbrace{aaa \dots a}_{n \text{ times}} \underbrace{bbb \dots b}_{n \text{ times}}$$

Then by the pumping lemma, there is some non-empty string y and strings x and z such that the FSA accepts $x \underbrace{yyy \dots y}_n z$ for every n . y can be either an unbroken string of a 's, or an unbroken string of b 's, or a string of a 's, followed by an unbroken string of b 's.

Say y is an unbroken string of a 's. xyz must be accepted by the FSA so xyz has the same number of a 's as b 's, but then $xyyz$ has more a 's and no more b 's, so $xyyz$ has more a 's than b 's, so it is not accepted by the FSA. Contradiction!

Say y is an unbroken string of b 's. xyz must be accepted by the FSA so xyz has the same number of a 's as b 's, but then $xyyz$ has more b 's and no more a 's, so $xyyz$ has more b 's than a 's, so it is not accepted by the FSA. Contradiction!

If y has a string of a 's then a string of b 's, then $xyyz$ will have a string of b 's preceding a string of a 's, so the FSA will not accept it. Contradiction!

In each case, the existence of such an FSA contradicts the pumping lemma.

2. a) Prove (with a derivation) that $\{P \vee Q, P \rightarrow R, R \rightarrow W, Q \rightarrow (R \wedge R)\} \vdash W$

1	$P \vee Q$	hyp
2	$P \rightarrow R$	hyp
3	$R \rightarrow W$	hyp
4	$Q \rightarrow (R \wedge R)$	hyp
5	<div style="border-left: 1px solid black; padding-left: 5px;">P</div>	hyp
6	<div style="border-left: 1px solid black; padding-left: 5px;">R</div>	MP 2,5
7	<div style="border-left: 1px solid black; padding-left: 5px;">W</div>	MP 3,6
8	<div style="border-left: 1px solid black; padding-left: 5px;">Q</div>	hyp
9	<div style="border-left: 1px solid black; padding-left: 5px;">$R \wedge R$</div>	MP 4,8
10	R	conj. elim 9
11	W	MP 3,10
12	W	disj. elim. 1 5-7 8-11

b) The soundness theorem (metatheorem 1) allows you to conclude - from the fact established in a) - that $\{P \vee Q, P \rightarrow R, R \rightarrow W, Q \rightarrow (R \wedge R)\} \Vdash W$

3. i) Give a categorical derivation of $\neg(\neg A \vee B) \rightarrow (A \wedge \neg B)$

1	$\neg(\neg A \vee B)$	hyp
2	$\neg A$	hyp
3	$\neg A \vee B$	Disj intro 2
4	$\neg(\neg A \vee B)$	reit 1
5	$\neg\neg A$	neg intro 2-4
6	A	neg. elim. 5
7	B	hyp
8	$\neg A \vee B$	disj intro 6
9	$\neg(\neg A \vee B)$	reit 1
10	$\neg B$	conj. elim 9
11	$A \wedge \neg B$	conj intro 6, 10
12	$\neg(\neg A \vee B) \rightarrow (A \wedge \neg B)$	cond. intro 1-11

ii) Give a derivation of $W \vee R$ from $\{(P \rightarrow P) \rightarrow (A \wedge \neg A), Q \rightarrow W, Q \rightarrow R\}$

The trick for this one is just to ignore every premise except $(P \rightarrow P) \rightarrow (A \wedge \neg A)$

1	$(P \rightarrow P) \rightarrow (A \wedge \neg A)$	hyp
2	$Q \rightarrow W$	hyp
3	$Q \rightarrow R$	hyp
4	$\neg(W \vee R)$	hyp
5	P	hyp
6	P	reit 5
7	$P \rightarrow P$	cond intro 5-6
8	$A \wedge \neg A$	MP 1,7
9	A	conj elim 8
10	$\neg A$	conj. elim 8
11	$\neg\neg(W \vee R)$	neg intro 4-10
12	$W \vee R$	neg elim 11

4. Prove the following statements, or give a counter-example:

a) If $\Gamma \vdash P$ and $\Delta \vdash Q$ then $\Gamma \cup \Delta \vdash P \wedge Q$

True. Say you have a derivation of P from the assumptions in Γ . Add to this derivation the assumptions from Δ , and derive Q . This longer derivation will contain a derivation of P and a derivation of Q , with undischarged assumptions $\Gamma \cup \Delta$. The last line will be a conjunction introduction yielding the conclusion $P \wedge Q$.

b) If $\Gamma \Vdash P$ and $\Delta \Vdash Q$ then $\Gamma \cup \Delta \Vdash P \wedge Q$

True. Say that some interpretation I simultaneously satisfies $\Gamma \cup \Delta$. Since it makes every sentence in Γ true, it makes P true. Since it makes every sentence in Δ true, it makes Q true. Since it makes P and Q true, then by the truth-table for \wedge , it makes $P \wedge Q$ true.

c) If Γ is satisfiable, then $\Gamma \cup \{P \vee \neg P\}$ is satisfiable.

True. Say you have an interpretation I making every sentence in Γ true. There must be such an interpretation because Γ is simultaneously satisfiable. I must also satisfy $P \vee \neg P$, since it is a logical truth. So I satisfies $\Gamma \cup \{P \vee \neg P\}$ and hence $\Gamma \cup \{P \vee \neg P\}$ is satisfiable.

d) If Γ is satisfiable, and Δ is satisfiable, then $\Gamma \cup \Delta$ is satisfiable.

False. Let $\Gamma = \{P\}$ and $\Delta = \{\neg P\}$. $\Gamma \cup \Delta = \{P, \neg P\}$, which is not satisfiable.

e) If $\Gamma \Vdash P$ then $\Gamma \cup \{\neg P\}$ is satisfiable.

False. Say that $\Gamma = \{P\}$. Then $\Gamma \Vdash P$ but $\Gamma \cup \{\neg P\} = \{P, \neg P\}$, which is not satisfiable.

f) If Γ is satisfiable, then $\{\neg S \mid S \in \Gamma\}$ is satisfiable.

False. Let $\Gamma = \{P \vee \neg P\}$. Then Γ is satisfiable, but $\{\neg S \mid S \in \Gamma\}$ is not.

5. We can proceed by induction on the length of derivations.

Base Case: Say that A is proven by a derivation of one line. Then A must be a hypothesis, and so the derivation establishes (the trivial fact) that $\{A\} \vdash A$. Trivially any truth assignment making every sentence in $\{A\}$ true makes A true, so $\{A\} \Vdash A$.

Induction step: Say that for any sentence A and set of sentences Γ , whenever there is a derivation of length k or less, of A from hypotheses Γ , then $\Gamma \Vdash A$. Say that $\Gamma^* \vdash A^*$ via a derivation of exactly $k + 1$ lines. If A^* is a hypothesis, then the entire proof must consist of hypotheses, since the hypotheses are all to be at the beginning of the proof, and $A^* \in \Gamma^*$. Hence, any truth assignment making every sentence in Γ^* true makes A^* true, so $\Gamma^* \Vdash A^*$. If A^* is not a hypothesis, it must follow by modus

ponens from two other sentences $B \rightarrow A^*$ and B , where $\Gamma_1 \vdash B \rightarrow A^*$, and $\Gamma_2 \vdash B$. ($\Gamma_1 \cup \Gamma_2 = \Gamma^*$; the point is that you don't need to use all the sentences in Γ^* for each of the premises, even if you use all of Γ^* to derive A^* .) Since the derivations of $B \rightarrow A^*$ and B are k lines or shorter, the induction hypothesis applies: from $\Gamma_1 \vdash B \rightarrow A^*$ we can infer $\Gamma_1 \Vdash B \rightarrow A^*$ and from $\Gamma_2 \vdash B$ we can infer $\Gamma_2 \Vdash B$. What we want to show is $\Gamma^* \Vdash A^*$. Say we have a truth-assignment I making every sentence in Γ^* true. Since $\Gamma_1 \subseteq \Gamma^*$, I makes every sentence in Γ_1 true, and so it makes $B \rightarrow A^*$ true. Since $\Gamma_2 \subseteq \Gamma^*$, I makes every sentence in Γ_2 true, so it makes B true. By the truth-table for \rightarrow , any truth assignment making B and $B \rightarrow A^*$ true must also make A^* true, so I makes A^* true. Thus $\Gamma^* \Vdash A^*$

6. (a) Say that Γ is a V-saturated set and $(\neg P \vee \neg Q) \in \Gamma$. Explain how we know that either $P \notin \Gamma$ or $Q \notin \Gamma$.

Say that $P \in \Gamma$ and $Q \in \Gamma$. In this case we can derive R and $\neg R$ from Γ , so Γ is inconsistent, which contradicts the V-saturatedness of Γ . So either $P \notin \Gamma$ or $Q \notin \Gamma$.

Here is how you derive R and $\neg R$ from $\{P, Q, \neg P \vee \neg Q\}$:

1	$\neg P \vee \neg Q$	hyp
2	P	hyp
3	Q	hyp
4	$\neg P$	hyp
5	R	hyp
6	P	reit 2
7	$\neg P$	reit 4
8	$\neg R$	neg intro 5-7
9	$\neg Q$	hyp
10	R	hyp
11	Q	reit 3
12	$\neg Q$	reit 9
13	$\neg R$	neg intro 10-12
14	$\neg R$	disj. elim. 1, 4-8, 9-13

1	$\neg P \vee \neg Q$	hyp
2	P	hyp
3	Q	hyp
4	$\neg P$	hyp
5	$\neg R$	hyp
6	P	reit 2
7	$\neg P$	reit 4
8	$\neg\neg R$	neg intro 5-7
9	$\neg Q$	hyp
10	$\neg R$	hyp
11	Q	reit 3
12	$\neg Q$	reit 9
13	$\neg\neg R$	neg intro 10-12
14	$\neg\neg R$	disj. elim. 1, 4 - 8, 9 - 13
15	R	neg. elim. 14

(b) This one you should look up!

1	$\forall x (P(x) \rightarrow Q(x))$	hyp
2	$\forall x P(x)$	hyp
3	x $\forall x (P(x) \rightarrow Q(x))$	1, reit
4	$\forall x P(x)$	2, reit
5	$P(x) \rightarrow Q(x)$	3, uq elim
6	$P(x)$	4, uq elim
7	$Q(x)$	5,6, cond elim
8	$\forall x Q(x)$	7, uq int
9	$\forall x P(x) \rightarrow \forall x Q(x)$	2-8, cond int

- (b) Is it possible for a relation to be symmetric and transitive, but irreflexive? If yes, give an example. If no, explain why not.

The answer is: yes, it can happen, though the only way it can happen is if nothing bears the relation to anything else. So, for example, the relation of “not being identical to oneself” is (trivially) symmetric and (trivially) transitive and irreflexive.

You would get partial credit if you (tacitly) assumed that there was at least one a and b such that Rab and then argued that R would have to be reflexive if it were symmetric and transitive.