

Solutions to Homework Assignment 3

1. (Recursively define a-palindrome and prove by induction that a-palindromes have even numbers of b's and c's.)

Definition: Base case: "a" and "aa" are a-palindromes.

Recursion clause: If σ is an a-palindrome, then $a\sigma a$, $b\sigma b$ and $c\sigma c$ are a-palindromes.

Proof: Say that σ is an a-palindrome. Proceed by induction on the number of letters in σ . (Let n be the number of letters in σ .)

Base case: It is more straightforward if we consider the base case for $n = 1$ or 2 . If the length of σ is 1 or 2 , then σ must be either "a" or "aa". Since neither of "a" or "aa" has any b's or c's in it, they trivially have an even number of b's and c's, namely zero.

Induction step: Say that any a-palindrome with k or fewer symbols has an even number of b's and c's. (Induction hypothesis)

Say that σ is an a-palindrome with $k + 1$ letters. We can assume $k > 1$ since we took care of 1 and 2 in the base case, so we can assume that σ has more than two letters. This can only happen as a result of an application of the recursion clause of the definition of a-palindrome, so we know that σ is one of: $a\sigma'a$, $b\sigma'b$ or $c\sigma'c$, where σ' is an a-palindrome with $k - 1$ symbols. The inductive hypothesis applies, so we know that σ' has an even number of b's and c's. Since σ has exactly the letters of σ' except that it has either exactly two more a's or exactly two more b's or exactly two more c's we know that σ must also have an even number of b's and c's.

2. (Show by induction that a $2^n \times 2^n$ chessboard with one square missing can be covered by L-tiles without overlap.)

Base case: Say that $n=2$. This is immediate, since whatever square is omitted, there will be an L-shape left over, as in this case (chosen square in red, remainder in blue):

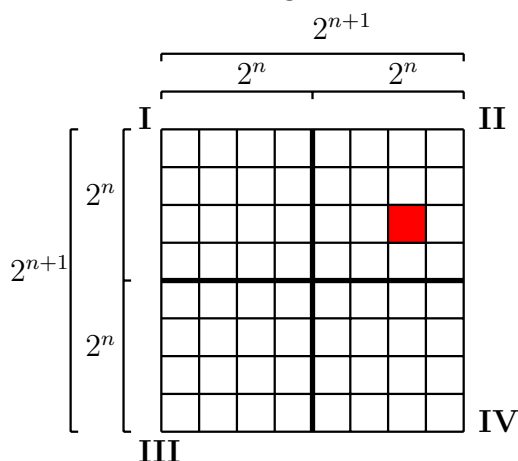


Induction step: Assume (induction hypothesis) that any $2^n \times 2^n$ board with one square removed can be covered without overlap with L-tiles. We want to show that any

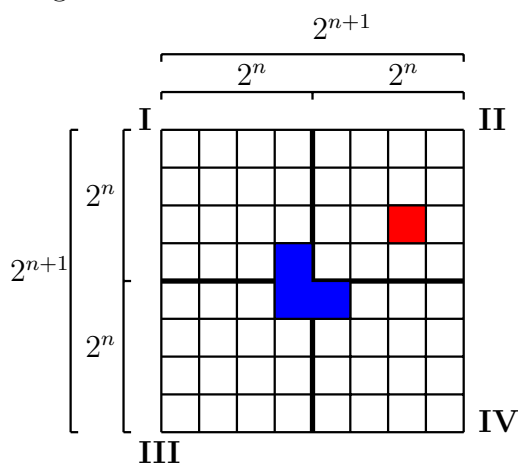
$2^{n+1} \times 2^{n+1}$ board with one square removed can be covered without overlap with L-tiles.

Proof of induction step:

Given a $2^{n+1} \times 2^{n+1}$ board with one square removed, break it into four $2^n \times 2^n$ chessboards as in the diagram:



By dividing the $2^{n+1} \times 2^{n+1}$ board into 4 quadrants, each of size $2^n \times 2^n$ we get a situation where the inductive hypothesis can help us. The red square will be in one of the quadrants - in the diagram above, it is in the upper right hand quadrant, labelled II. By the induction hypothesis we know that that quadrant II can be covered in L-tiles, leaving the red square uncovered. To address the other three quadrants, we need to change them from chessboards to chessboards with one square missing. Here's the trick: Place one L-tile in the center so it covers one square of each quadrant, as in this diagram:



Now we have a situation where we can apply the induction hypothesis to the other 3 quadrants: Each of them is a $2^n \times 2^n$ chessboard with one square missing. Hence we can cover each of the remaining quadrants with L-tiles. The tiling of all four quadrants, plus the tile placed in the center, cover the whole $2^{n+1} \times 2^{n+1}$ board except for the missing square. This proves the induction hypothesis.

From the base case and the induction step, we conclude that the thesis is true in general: every $2^{n+1} \times 2^{n+1}$ board with one square missing can be covered without overlap with L-tiles.

3. (Prove by induction: For every n greater than 3, $2^n < n! < n^n$)

Base case: (The base case will be $n = 4$ since we are restricting attention to $n > 3$.) If $n = 4$ then $2^n = 2^4 = 16$, $n! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, and $n^n = 4^4 = 256$. $16 < 24 < 256$, so the base case holds.

Induction Step: Assume that for k , $k > 4$, $2^k < k! < k^k$. $2^{k+1} = 2 \cdot 2^k$ and $(k+1)! = (k+1) \cdot k!$. Since $2 < k+1$ by hypothesis, and $2^k < k!$ we have one of the inequalities we want to prove: $2^{k+1} < (k+1)!$, by the elementary principle that if $a < b$ and $c < d$ with a, b, c, d all positive integers, then $a \cdot c < b \cdot d$.

Similarly, since $k+1$ is positive and greater than one, $k^k < (k+1)^k$, so this inequality holds:

$(k+1)! = (k+1) \cdot k! < (k+1) \cdot k^k < (k+1) \cdot (k+1)^k = (k+1)^{k+1}$. Taking the first and last terms of this inequality gives us the other inequality we want. Hence $2^{k+1} < (k+1)! < (k+1)^{k+1}$.

This proves the induction step, and so it completes the proof.

4. (Say that in addition to the nickel, we had a 3 cent coin. Prove by induction that every amount of money above 7 cents could be made up of some collection of 3 cent coins and 5 cent coins.)

Proof:

Say that we have some amount n of money.

Base case: The base case here is a bit unusual, because we need to consider three possibilities: $n = 8$ cents, $n = 9$ cents and $n = 10$ cents. (We could just treat 8 cents, but we would discover, when we turned to cover the induction step, that we would have to cover a couple of irritating exceptions. So treating all three of these simplifies the proof overall.) $8 = 5 + 3$, $9 = 3 + 3 + 3$, $10 = 5 + 5$.

Induction Step: Say that we have a number $k > 10$. Induction hypothesis: assume that for every n , with $8 \leq n < k$, n cents can be made up of a combination of 3 and 5 cent coins. $k - 3$ cents will be covered by the induction hypothesis. (This is why we needed to treat 8, 9 and 10 in the base case. If we had only treated 8, then we would only know that $k > 8$, so $k - 3$ could be 7 or 6, which wouldn't be covered by the induction hypothesis.) Say that $k - 3$ cents is made up of l three cent coins and m five cent coins. Then k cents is made up of $l + 1$ three cent coins and m five cent coins.

That completes the induction hypothesis and hence the proof.