Philosophy 303
Symbolic Logic
Winter, 2009

Solutions to Homework Assignment 2

1. (Chapter 2, Exercise 35, Part b)

   bb, ab, abb

2. Chapter 2, Exercise 36 b) and d)

   b) (There are many possibilities. Here are five). a, b, aab, bab, aabab.

   d) Note that the only string in $(\{a\}^* \cap \{b\}^*)$ is the empty string. So we are really asking for strings in $\{ab\}^*$. Here are five: ϵ, ab, abab, ababab, abababab.

3. (Chapter 2, Exercise 39.)

   Let $X = \{\epsilon, aa\}$, $Y = \{aa\}$, $Z = \{aaaa\}$.
   Then $X \circ (Y \cap Z) = \{\epsilon, aa\} \circ \emptyset = \emptyset$.
   But $(X \circ Y) \cap (X \circ Z) = \{aa, aaaa\} \cap \{aaaa, aaaaaa\}$.
   $= \{aaaa\}$.

4. Chapter 2 Exercise 49 Part c) and e)

   c) Here are the first six values of the function with $f(0) = 1$ and $f(n+1) = f(n)$:
   1,1,1,1,1,1

   e) Here are the first six values of the function with $f(0) = 1$ and $f(n+1) = 2 + (n \cdot (f(n) - 1))$:
   2, 3, 4, 8, 23, 90

5. Answer: Base clause: $f(0) = a^0 = 1$; Recursion clause: $f(n+1) = a^{(n+1)} = a^n \cdot a$

6. Chapter 2 Exercise 52

   Base case
   The only set with no members is $\emptyset$, $\mathcal{P}(\emptyset) = \{\emptyset\}$ has 1 member. This is what we need to prove, since $2^0 = 1$.

   Induction Step
   Assume that the thesis is true for every set X with n or fewer members. Say that X'

has $n+1$ members. Then X' isn't empty, so we can pick some element $a \in X'$. Let $X'' = X' - \{a\}$. We know that X'' has n members, so the induction hypothesis applies to X'': $\mathcal{P}(X'')$ has 2^n members. Every subset of X'' is a subset of X', since $X'' \subseteq X'$, and so $\mathcal{P}(X'') \subseteq \mathcal{P}(X')$.

Note that we can pair up all the elements of $\mathcal{P}(X'')$ with elements of $\mathcal{P}(X') - \mathcal{P}(X'')$ so that every member of $\mathcal{P}(X'')$ corresponds to exactly one member of $\mathcal{P}(X') - \mathcal{P}(X'')$: for any $Y \in \mathcal{P}(X'')$, let its partner be $\{a\} \cup Y$, which will be a member of $\mathcal{P}(X') - \mathcal{P}(X'')$. This shows that there are the same number of elements in $\mathcal{P}(X'')$ and in $\mathcal{P}(X') - \mathcal{P}(X'')$. But we already know that $\mathcal{P}(X'')$ has 2^n elements, so $\mathcal{P}(X') - \mathcal{P}(X'')$ has 2^n elements as well.

Finally, we note that $\mathcal{P}(X') = \mathcal{P}(X'') \cup (\mathcal{P}(X') - \mathcal{P}(X''))$. Also, $\mathcal{P}(X'')$ and $\mathcal{P}(X') - \mathcal{P}(X'')$ have no elements in common. (This point is crucial.) So the size of $\mathcal{P}(X') =$ (size of $\mathcal{P}(X'')$) $+ (sizeof(\mathcal{P}(X') - \mathcal{P}(X'')) = 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$.

This proves the induction step.

The base case and the induction step together prove the claim.

7. Chapter 2 Exercise 58

   Inductive Definition of X^*: **Base Clause**: $\epsilon \in X^*$
   Recursion Clause: If $x \in X^*$ and $y \in X$ then $xy \in X^*$
   Closure Clause: Nothing else is in X^* (It's ok if you don't make the closure clause explicit)

   Proof that this defines the Kleene *: Let $X^{Inductive*}$ be the set we've defined, and let X^* be the Kleene * of X according to the textbook definition. (i.e. $X^* = \{x_1 \ldots x_n \,/\, n \geq 0 \text{ and } x_1, \ldots x_n \in X\}$. We want to prove that $X^{Inductive*} = X^*$. As usual, we prove containment both ways.

   $\Rightarrow$ Say that $s \in X^{Inductive*}$. We want to show that $s \in X^*$. We can proceed by induction (on the number of symbols in s).
   First note (base case) that if $s = \epsilon$, then $s \in X^*$ by definition.
   Assume (induction hypothesis) that s is not the empty string, and that for every string with fewer letters than s, if $s \in X^{Inductive*}$ then $s \in X^*$. From the inductive definition, we can see that there must be some $s' \in X^{Inductive*}$ and $\tilde{s} \in X$ such that $s = s'\tilde{s}$. Since s' is shorter than s, we know that $s' \in X^*$ by the induction hypothesis. Hence by the definition of X^* there are strings $x_1 \in X, x_2 \in X, \ldots, x_i \in X$ such that $s' = x_1 x_2 \ldots x_i$. But then $s = x_1 x_2 \ldots x_i \tilde{s}$. That is, s consists of a concatenation of strings from X, so by definition s is in X^*. This completes the inductive step.

   The base step and the induction step allow us to conclude $s \in X^*$. Since s was chosen arbitrarily, this proves that $X^{Inductive*} \subseteq X^*$.

$\Leftarrow$ Say that $s \in X^*$. We want to show that $s \in X^{Inductive}*$. Again we proceed by induction (on the number of symbols in s).

First note (base case) that if $s = \epsilon$, then $s \in X^{Inductive*}$ by definition.
Assume (induction hypothesis) that s is not the empty string, and that for every string with fewer letters than s, if $s \in X^*$ then $s \in X^{Inductive}*$. By the definition of X^*, there are strings $x_1 \in X, x_2 \in X, \ldots, x_{l-1} \in X, x_l \in X$ such that $s = x_1x_2 \ldots x_{l-1}x_l$. By the definition of X^*, $s' = x_1x_2 \ldots x_{l-1} \in X^*$, so by the inductive hypothesis, $s' = x_1x_2 \ldots x_{l-1} \in X^{Inductive*}$. By the inductive definition of $X^{Inductive*}$, $s'x_l \in X^{Inductive*}$. Since $s = s'x_l$, we have that $s \in X^{Inductive*}$. This completes the inductive step.

The base step and the induction step allow us to conclude $s \in X^{Inductive*}$. Since s was chosen arbitrarily, this proves that $X^* \subseteq X^{Inductive*}$.

Combining $\Leftarrow$ and $\Rightarrow$, along with the second version of the principle of extensionality, gives: $X^* = X^{Inductive*}$