Philosophy 303
Symbolic Logic
Winter, 2009

Solutions to Homework Assignment 1

1. Chapter 2, Exercise 2, Parts b), e) and f).

   b) $(x+2)^2 = x^2 + 4x + 4$

   e) If John gave some money to x and Mary gave some money to y (with $x \neq y$) then y may not have wanted x to know that Andrea gave y additional money.

   f) $X \cap X = X$

2. (Chapter 2, Exercise 3 c), d) and i)

   c) $\{\}$ (Also acceptable: $\emptyset$)

   d) $\{1, 2\}$

   i) $\{1, 2, 3\}$

3. Chapter 2 exercise 5 b) and e)

   b) We prove both directions:
   a) Say that $W \in X \cap (Y \cap Z)$
   Then: $W \in X$ and $W \in Y \cap Z$
   So: i) $W \in X$ and ii) $W \in Y$ and iii) $W \in Z$
   By i) and ii) we have: $W \in X \cap Y$
   Combining the preceding line with iii) we have: $W \in (X \cap Y) \cap Z$
   W was an arbitrary choice, so we can conclude that $X \cap (Y \cap Z) \subseteq (X \cap Y) \cap Z$
   This proves one direction.

   b) Say that $W \in (X \cap Y) \cap Z)$
   Then: $W \in Z$ and $W \in X \cap Y$
   So: i) $W \in X$ and ii) $W \in Y$ and iii) $W \in Z$
   By ii) and iii) we have: $W \in Y \cap Z$
   Combining the preceding line with i) we have: $W \in (X \cap Y) \cap Z$
   W was an arbitrary choice, so we can conclude that $(X \cap Y) \cap Z \subseteq X \cap (Y \cap Z)$
   This proves the other direction, so we can conclude that:
   $(X \cap Y) \cap Z = X \cap (Y \cap Z)$

e) Prove that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

This is a set identity, and as so often it is easiest to "prove both directions".

First, we prove $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$

Suppose $W \in X \cup (Y \cap Z)$. Then $W \in X$ *or* $W \in (Y \cap Z)$

a) Say $W \in X$. Weakening, we have that $W \in X$ *or* $W \in Y$. Also by weakening, we have $W \in X$ *or* $W \in Z$.
So we have $W \in X \cup Y$ and $W \in X \cup Z$

Hence, $W \in (X \cup Y) \cap (X \cup Z)$

b) Say that $W \in (Y \cap Z)$
Then $W \in Y$ *and* $W \in Z$
Weakening twice, we have $W \in X$ or $W \in Y$ *and* $W \in X$ or $W \in Z$
That is, $W \in X \cup Y$ and $W \in X \cup Z$
Hence, $W \in (X \cup Y) \cap (X \cup Z)$

One of a) or b) must be true, and on either one, $W \in (X \cup Y) \cap (X \cup Z)$. We conclude $W \in (X \cap Y) \cup (X \cap Z)$. This gives us the first direction: $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$

Second, we prove $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$

Suppose $W \in (X \cup Y) \cap (X \cup Z)$. Then i) $W \in (X \cup Y)$ *and* ii) $W \in (X \cup Z)$

There are two possibilities: i) $W \in X$ or ii) $W \notin X$.
If i) then by weakening $W \in X$ or $W \in Y \cap Z$.
Hence $W \in X \cup (Y \cap Z)$.

If ii) then we have $W \notin X$ and $W \in X \cup Y$. That is: α) $W \notin X$ and β) either $W \in X$ or $W \in Y$. Combining α) and β) with commonsense logic that we will start analyzing in chapter IV, we have $W \in Y$.

Also, if ii)we have $W \notin X$ and $W \in X \cup Z$. That is: α) $W \notin X$ and β) either $W \in X$ or $W \in Z$. Combining α) and β) with commonsense logic that we will start analyzing in chapter IV, we have $W \in Z$.

So, if ii) we have $W \in Y$ and $W \in Z$.
Hence, if ii) then $W \in Y \cap Z$, so by weakening, $W \in X$ or $W \in Y \cap Z$.

Hence if ii) then $W \in X \cup (Y \cap Z)$.

One of i) or ii) must be true, and on either one, $W \in X \cup (Y \cap Z)$ so we conclude $W \in X \cup (Y \cap Z)$. This proves the second direction.

Combining both directions, we can conclude $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

4. Chapter 2, Exercise 7 b)

We want to show that for all sets X and Y, $Y \subseteq X$ if and only if $X \cup Y = X$. We prove the implication in both directions.

First, suppose that $Y \subseteq X$. We'll prove $X \cup Y = X$ by proving containment in both directions. $X \subseteq (X \cup Y)$ is trivial (make sure you see why and how this is so), so we only need to prove $X \cup Y \subseteq X$.
Say $Z \in X \cup Y$. Then either i) $Z \in X$ or ii) $Z \in Y$. If i), then we already have $Z \in X$. If ii), then since $Z \in Y$ and $Y \subseteq X$, we have that $Z \in X$. Since one of i) and ii) must be true, and either way $Z \in X$, we can conclude $Z \in X$. Hence $X \cup Y \subseteq X$, and so $X \cap Y = X$ since we know both directions of containment.

To prove the second direction, it is particularly easy to prove the *Contrapositive*.
That is, if we prove that "If not - P then not - Q" we can conclude "If Q then P" is true.
(Make sure that you understand this inference!).
Say that $Y \not\subseteq X$. Then there must be some $Z \in Y$, such that $Z \notin X$.
But then by weakening, $Z \in Y$ or $Z \in X$ and $Z \notin X$.
By the definition of union, $Z \in Y \cup X$ and $Z \notin X$.
Hence $Y \cup X \not\subseteq X$, so it isn't true that $Y \cup X = X$.
That is, if $Y \not\subseteq X$ then it isn't true that $Y \cup X = X$.
This is the contrapositive of "If $Y \cup X = X$ then $Y \subseteq X$".
This proves the second direction of the implication.

Since we have both directions, we conclude $Y \subseteq X$ if and only if $X \cup Y = X$.

Note: In general, there are many correct answers for the following questions.

5. (Chapter 2, Exercise 26, Parts a), d), h).)

a) $aabb$ $ababb$ $aaaabb$
$aaaaabb$ $abbabb$

d) ϵ aa $abab$
$abbabb$ $bbbb$

h) $aaaaa$ $bbabb$ $bbaaa$ That's all the possibilities.
$aaabb$

6. (Chapter 2, Exercise 27, Part c)

c)

aaa	$u := a, x := a, y := a$
a	$u := a, x := \epsilon, y := \epsilon$
aa	$u := a, x := \epsilon, y := a$
$baabb$	$u := b, x := \epsilon, y := aabb$
$abaab$	$u := a, x := ba, y := ab$

7. (Chapter 2, Exercise 28, Parts b), d), and f).)

b) $axayaz$

d) $xaby$

f) uu'