

Solutions to Homework Assignment 1

1. (Chapter 2 Exercise 1)

A word is ambiguous if it has more than one meaning. So for example “bank” is ambiguous, because it can mean the side of a river or the place where people deposit money, among other things. However, calling a financial institution a bank is not providing a vague description, and this description may be accurate.

A word is vague if for one or more cases (“borderline cases”) it is not clear whether the word applies or does not apply to that case. Classic examples are “heap”, “bald” and “poor”: for each of these there are borderline cases that don’t clearly count or fail to count. The description of a man with no hair on his head as “bald” can be accurate and unambiguous, but uses vague terminology.

A description is inaccurate if it describes something as being different from how it actually is. Saying that Los Angeles is 100 kilometers driving distance from Chicago is precise and unambiguous, but inaccurate.

2. Chapter 2, Exercise 2, Parts b) e) and f).

b) $(x + 2)^2 = x^2 + 4x + 4$

e) If John gave some money to x and Mary gave some money to y (with $x \neq y$) then y may not have wanted x to know that Andrea gave y additional money.

f) $X \cap X = X$

3. (Chapter 2, Exercise 3 c) d) and i)

c) $\{\}$ (Also acceptable: \emptyset)

d) $\{1, 2\}$

i) $\{1, 2, 3\}$

4. Chapter 2 exercise 5 b) and e)

b) We prove both directions:

a) Say that $W \in X \cap (Y \cap Z)$

Then: $W \in X$ and $W \in Y \cap Z$

So: i) $W \in X$ and ii) $W \in Y$ and iii) $W \in Z$

By i) and ii) we have: $W \in X \cap Y$

Combining the preceding line with iii) we have: $W \in (X \cap Y) \cap Z$

W was an arbitrary choice, so we can conclude that $X \cap (Y \cap Z) \subseteq (X \cap Y) \cap Z$

This proves one direction.

b) Say that $W \in (X \cap Y) \cap Z$

Then: $W \in Z$ and $W \in X \cap Y$

So: i) $W \in X$ and ii) $W \in Y$ and iii) $W \in Z$

By ii) and iii) we have: $W \in Y \cap Z$

Combining the preceding line with i) we have: $W \in (X \cap Y) \cap Z$

W was an arbitrary choice, so we can conclude that $(X \cap Y) \cap Z \subseteq X \cap (Y \cap Z)$

This proves the other direction, so we can conclude that:

$(X \cap Y) \cap Z = X \cap (Y \cap Z)$

e) Prove that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

This is a set identity, and as so often it is easiest to “prove both directions”.

First, we prove $X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$

Suppose $W \in X \cup (Y \cap Z)$. Then $W \in X$ or $W \in (Y \cap Z)$

a) Say $W \in X$. Weakening, we have that $W \in X$ or $W \in Y$. Also by weakening, we have $W \in X$ or $W \in Z$.

So we have $W \in X \cup Y$ and $W \in X \cup Z$

Hence, $W \in (X \cup Y) \cap (X \cup Z)$

b) Say that $W \in (Y \cap Z)$

Then $W \in Y$ and $W \in Z$

Weakening twice, we have $W \in X$ or $W \in Y$ and $W \in X$ or $W \in Z$

That is, $W \in X \cup Y$ and $W \in X \cup Z$

Hence, $W \in (X \cup Y) \cap (X \cup Z)$

One of a) or b) must be true, and on either one, $W \in (X \cup Y) \cap (X \cup Z)$.

We conclude $W \in (X \cup Y) \cap (X \cup Z)$. This gives us the first direction:

$X \cup (Y \cap Z) \subseteq (X \cup Y) \cap (X \cup Z)$

Second, we prove $(X \cup Y) \cap (X \cup Z) \subseteq X \cup (Y \cap Z)$

Suppose $W \in (X \cup Y) \cap (X \cup Z)$. Then i) $W \in (X \cup Y)$ and ii) $W \in (X \cup Z)$

There are two possibilities: i) $W \in X$ or ii) $W \notin X$.

If i) then by weakening $W \in X$ or $W \in Y \cap Z$.

Hence $W \in X \cup (Y \cap Z)$.

If ii) then we have $W \notin X$ and $W \in X \cup Y$. That is: α) $W \notin X$ and β) either $W \in X$ or $W \in Y$. Combining α) and β) with commonsense logic that we will start analyzing in chapter IV, we have $W \in Y$.

Also, if ii) we have $W \notin X$ and $W \in X \cup Z$. That is: α) $W \notin X$ and β) either $W \in X$ or $W \in Z$. Combining α) and β) with commonsense logic that we will start analyzing in chapter IV, we have $W \in Z$.

So, if ii) we have $W \in Y$ and $W \in Z$.

Hence, if ii) then $W \in Y \cap Z$, so by weakening, $W \in X$ or $W \in Y \cap Z$.

Hence if ii) then $W \in X \cup (Y \cap Z)$.

One of i) or ii) must be true, and on either one, $W \in X \cup (Y \cap Z)$ so we conclude $W \in X \cup (Y \cap Z)$. This proves the second direction.

Combining both directions, we can conclude $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

5. Chapter 2, Exercise 7 b)

We want to show that for all sets X and Y , $Y \subseteq X$ if and only if $X \cup Y = X$. We prove the implication in both directions.

First, suppose that $Y \subseteq X$. We'll prove $X \cup Y = X$ by proving containment in both directions. $X \subseteq (X \cup Y)$ is trivial (make sure you see why and how this is so), so we only need to prove $X \cup Y \subseteq X$.

Say $Z \in X \cup Y$. Then either i) $Z \in X$ or ii) $Z \in Y$. If i), then we already have $Z \in X$. If ii), then since $Z \in Y$ and $Y \subseteq X$, we have that $Z \in X$. Since one of i) and ii) must be true, and either way $Z \in X$, we can conclude $Z \in X$. Hence $X \cup Y \subseteq X$, and so $X \cap Y = X$ since we know both directions of containment.

To prove the second direction, it is particularly easy to prove the *Contrapositive*.

That is, if we prove that "If not - P then not - Q" we can conclude "If Q then P" is true.

(Make sure that you understand this inference!).

Say that $Y \not\subseteq X$. Then there must be some $Z \in Y$, such that $Z \notin X$.

But then by weakening, $Z \in Y$ or $Z \in X$ and $Z \notin X$.

By the definition of union, $Z \in Y \cup X$ and $Z \notin X$.

Hence $Y \cup X \not\subseteq X$, so it isn't true that $Y \cup X = X$.

That is, if $Y \not\subseteq X$ then it isn't true that $Y \cup X = X$.

This is the contrapositive of "If $Y \cup X = X$ then $Y \subseteq X$ ".

This proves the second direction of the implication.

Since we have both directions, we conclude $Y \subseteq X$ if and only if $X \cup Y = X$.

Note: In general, there are many correct answers for the following questions.

6. (Chapter 2, Exercise 26, Parts a), d), h).)

a) $aabb \quad ababb \quad aaaabb$
 $aaaaabb \quad abbabb$

d) $\epsilon \quad aa \quad abab$
 $abbabb \quad bbbb$

h) $aaaaa \quad bbabb \quad bbaaa$ That's all the possibilities.
 $aaabb$

7. (Chapter 2, Exercise 27, Part c)

c) $aaa \quad u := a, x := a, y := a$
 $a \quad u := a, x := \epsilon, y := \epsilon$
 $aa \quad u := a, x := \epsilon, y := a$
 $baabb \quad u := b, x := \epsilon, y := aabb$
 $abaab \quad u := a, x := ba, y := ab$

8. (Chapter 2, Exercise 28, Parts b), d), and f).)

b) $axayaz$

d) $xaby$

f) uu'