

## Solutions to Homework Assignment 8

1. a1) Prove that you can derive  $\neg P \vee Q$  from  $P \rightarrow Q$ .

1	$P \rightarrow Q$	hyp
2	$\neg(\neg P \vee Q)$	hyp
3	$P$	hyp
4	$Q$	m p 1 3
5	$\neg P \vee Q$	disj intro 4
6	$\neg(\neg P \vee Q)$	reit 2
7	$\neg P$	neg intro 3-5
8	$\neg P \vee Q$	disj intro 7
9	$\neg(\neg P \vee Q)$	reit 2
10	$\neg\neg(\neg P \vee Q)$	neg intro 2-9
11	$\neg P \vee Q$	neg elim. 9

a2) Prove that you can derive  $P \rightarrow Q$  from  $\neg P \vee Q$ .

1	$\neg P \vee Q$	hyp
2	$P$	hyp
3	$\neg P$	hyp
4	$\neg Q$	hyp
5	$P$	Reit 2
6	$\neg P$	reit 3
7	$\neg\neg Q$	neg int 4-6
8	$Q$	neg elim 7
9	$Q$	hyp
10	$Q$	reit 9
11	$Q$	1, 3-8 9-10 disj. elim.
12	$P \rightarrow Q$	2-11 cond intro

2. b1) Prove that you can derive  $\neg(P \wedge \neg Q)$  from  $P \rightarrow Q$ .

1	$P \rightarrow Q$	hyp
2	$(P \wedge \neg Q)$	hyp
3	$P$	conj elim. 2
4	$Q$	m p 1 3
5	$\neg Q$	conj elim 2
6	$\neg(P \wedge \neg Q)$	neg intro 2-5

b2) Prove that you can derive  $P \rightarrow Q$  from  $\neg(P \wedge \neg Q)$ .

1	$\neg(P \wedge \neg Q)$	hyp
2	<div style="border-left: 1px solid black; padding-left: 5px; border-bottom: 1px solid black;"><math>P</math></div>	hyp
3	<div style="border-left: 1px solid black; padding-left: 5px; border-bottom: 1px solid black;"><math>\neg Q</math></div>	hyp
4	<div style="border-left: 1px solid black; padding-left: 5px;"><math>(P \wedge \neg Q)</math></div>	conj intro 2, 3
5	$\neg(P \wedge \neg Q)$	reit 1
6	$\neg\neg Q$	neg intro 2-5
7	$Q$	neg elim 6
8	$P \rightarrow Q$	cond intro 2-7

3. In each case there are many equivalent possibilities. I'll just give one for each question.

a)  $P \rightarrow (Q \rightarrow \neg R)$   
 $\neg P \vee (\neg Q \vee \neg R)$

b)  $(P \wedge Q) \vee (P \wedge R)$   
 $\neg(\neg P \vee \neg Q) \vee \neg(\neg P \vee \neg R)$

The truth tables you draw will depend on which sentences you choose as translations.

Here is the table for the answer given above for a):

P	Q	R	P	$\rightarrow$	(Q	$\rightarrow$	$\neg$	R)	$\neg$	P	$\vee$	( $\neg$	Q	$\vee$	$\neg$	R)
T	T	T	T	<b>F</b>	T	T	F	T	F	T	<b>F</b>	F	T	F	F	T
T	T	F	T	<b>T</b>	T	T	T	F	F	T	<b>T</b>	F	T	T	T	F
T	F	T	T	<b>T</b>	F	T	F	T	F	T	<b>T</b>	T	F	T	F	T
T	F	F	T	<b>T</b>	F	T	T	F	F	T	<b>T</b>	T	F	T	T	F
F	T	T	F	<b>T</b>	T	F	F	T	T	F	<b>T</b>	F	T	F	F	T
F	T	F	F	<b>T</b>	T	F	T	F	T	F	<b>T</b>	F	T	T	T	F
F	F	T	F	<b>T</b>	F	F	F	T	T	F	<b>T</b>	T	F	T	F	T
F	F	F	F	<b>T</b>	F	F	T	F	T	F	<b>T</b>	T	F	T	T	F

4. Prove, in the system that results from adding the symbol and rules for “neither... nor ...” to boolean propositional logic, that  $P \downarrow P$  is logically equivalent to  $\neg P$  and  $(P \downarrow Q) \downarrow (P \downarrow Q)$  is logically equivalent to  $P \vee Q$ . (Half of each equivalence was given in the question.)

1	$\neg P$	hyp
2	$\neg P$	reit 1
3	$P \downarrow P$	nn intro 1,2

The second part:

1	$P \vee Q$	hyp																		
2	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="padding-right: 10px;">3</td> <td style="border-left: 1px solid black; padding-left: 10px; border-bottom: 1px solid black;"><math>P</math></td> <td style="padding-left: 10px;">hyp 1</td> </tr> <tr> <td>4</td> <td style="border-left: 1px solid black; padding-left: 10px; border-bottom: 1px solid black;"> <table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="padding-right: 10px;">5</td> <td style="border-left: 1px solid black; padding-left: 10px;"><math>(P \downarrow Q)</math></td> <td>hyp</td> </tr> <tr> <td>6</td> <td style="border-left: 1px solid black; padding-left: 10px;"><math>P</math></td> <td>reit 2</td> </tr> <tr> <td>7</td> <td style="border-left: 1px solid black; padding-left: 10px;"><math>\neg P</math></td> <td>nn elim 3</td> </tr> </table> </td> <td style="padding-left: 10px;">neg intro 3-5</td> </tr> <tr> <td>8</td> <td style="border-left: 1px solid black; padding-left: 10px;"><math>\neg(P \downarrow Q)</math></td> <td>neg intro 3-5</td> </tr> </table>	3	$P$	hyp 1	4	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="padding-right: 10px;">5</td> <td style="border-left: 1px solid black; padding-left: 10px;"><math>(P \downarrow Q)</math></td> <td>hyp</td> </tr> <tr> <td>6</td> <td style="border-left: 1px solid black; padding-left: 10px;"><math>P</math></td> <td>reit 2</td> </tr> <tr> <td>7</td> <td style="border-left: 1px solid black; padding-left: 10px;"><math>\neg P</math></td> <td>nn elim 3</td> </tr> </table>	5	$(P \downarrow Q)$	hyp	6	$P$	reit 2	7	$\neg P$	nn elim 3	neg intro 3-5	8	$\neg(P \downarrow Q)$	neg intro 3-5	neg intro 5-7
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10	$Q$	hyp																		
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14	$\neg Q$	nn elim 8																		
15	$\neg(P \downarrow Q)$	neg intro 8-10																		
16	$\neg(P \downarrow Q)$	reit 12																		
17	$(P \downarrow Q) \downarrow (P \downarrow Q)$	nn intro 12,13																		

5. **Exercise 5:**

Given the rules of introduction and elimination for  $\downarrow$ , figure out what its truth-table looks like and write it down.

A	B	A	$\downarrow$	B
T	T	T	<b>F</b>	T
T	F	T	<b>F</b>	F
F	T	F	<b>F</b>	T
F	F	F	<b>T</b>	F

6. **Exercise 6i):**

Prove using the rules of chapter IV that these two sentences are logically equivalent:

$$P \wedge (Q \vee R) \text{ and } (P \wedge Q) \vee (P \wedge R)$$

1	$P \wedge (Q \vee R)$	hyp		
2	$P$	conj elim 1		
3	$Q \vee R$	conj elim 1		
4	<table style="border-collapse: collapse;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>Q</math></td> <td style="padding-left: 10px;">hyp</td> </tr> </table>	$Q$	hyp	hyp
$Q$	hyp			
5	<table style="border-collapse: collapse;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>(P \wedge Q)</math></td> <td style="padding-left: 10px;">conj intro 2 4</td> </tr> </table>	$(P \wedge Q)$	conj intro 2 4	conj intro 2 4
$(P \wedge Q)$	conj intro 2 4			
6	$(P \wedge Q) \vee (P \wedge R)$	disj intro 5		
7	<table style="border-collapse: collapse;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>R</math></td> <td style="padding-left: 10px;">hyp</td> </tr> </table>	$R$	hyp	hyp
$R$	hyp			
8	<table style="border-collapse: collapse;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>P \wedge R</math></td> <td style="padding-left: 10px;">conj intro 2 7</td> </tr> </table>	$P \wedge R$	conj intro 2 7	conj intro 2 7
$P \wedge R$	conj intro 2 7			
9	$(P \wedge Q) \vee (P \wedge R)$	disj intro 8		
10	$(P \wedge Q) \vee (P \wedge R)$	disj elim 1 4–6 7–9		

1	$(P \wedge Q) \vee (P \wedge R)$	hyp
2	$P \wedge Q$	hyp
3	$Q$	conj elim 2
4	$P$	conj elim 2
5	$Q \vee R$	disj intro 4
6	$P \wedge (Q \vee R)$	conj intro 4 5
7	$P \wedge R$	hyp
8	$R$	conj elim 8
9	$P$	conj elim 8
10	$Q \vee R$	disj intro 9
11	$P \wedge (Q \vee R)$	conj intro 9 10
12	$P \wedge (Q \vee R)$	disj elim 1 2–6 7–10

**Exercise 6ii):**

Then construct a truth-table to show that these two sentences are truth-functionally equivalent:

$$P \wedge (Q \vee R) \text{ and } (P \wedge Q) \vee (P \wedge R)$$

P	Q	R	P	$\wedge$	(Q	$\vee$	R)	(P	$\wedge$	Q)	$\vee$	(P	$\wedge$	R)
T	T	T	T	<b>T</b>	T	T	T	T	T	T	<b>T</b>	T	T	T
T	T	F	T	<b>T</b>	T	T	F	T	T	T	<b>T</b>	T	F	F
T	F	T	T	<b>T</b>	F	T	T	T	F	F	<b>T</b>	T	T	T
T	F	F	T	<b>F</b>	F	F	F	T	F	F	<b>F</b>	T	F	F
F	T	T	F	<b>F</b>	T	T	T	F	F	T	<b>F</b>	F	F	T
F	T	F	F	<b>F</b>	T	T	F	F	F	T	<b>F</b>	F	F	F
F	F	T	F	<b>F</b>	F	T	T	F	F	F	<b>F</b>	F	F	T
F	F	F	F	<b>F</b>	F	F	F	F	F	F	<b>F</b>	F	F	F