Philosophy 303
Symbolic Logic
Winter, 2009

Homework Assignment 8

Pay attention to all the comments interspersed among the problems: this is information you'll be expected to remember. In this problem set, I'll present a fact about logic, at the level of analysis we have been studying in Chapter 4. An interesting feature of the Boolean connectives ($\vee$, $\neg$, $\wedge$, . . .) is that just a pair of them suffice to define them all. In fact, we can define a single connective that suffices to define all the others! Here are the details:

1. First we need to help ourselves to a fact we'll prove later: The language with connectives { "$\vee$", "$\wedge$", and "$\neg$" } is powerful enough to define *all possible* Boolean connectives. (This proof will involve developing some conceptual tools, including clarifying what "all possible" means in this context.) With that fact in our pocket, we can reduce the number of connectives we need even further. Here are a few first steps in that direction:

   Exercise 1:
   a1) Prove that you can derive $\neg P \vee Q$ from $P \rightarrow Q$.
   a2) Prove that you can derive $P \rightarrow Q$ from $\neg P \vee Q$.

   We will say that two sentences are *logically equivalent* when each can be derived from the other. These two results show that any formula containing "$\rightarrow$" is logically equivalent, in our system, to some formula containing no occurrence of "$\rightarrow$". The symbol for "$\rightarrow$" can therefore be seen as just a convenient abbreviation in a language whose only connectives are "$\vee$", "$\wedge$", and "$\neg$".

2. Similarly, we can prove that "$\rightarrow$" can be eliminated in favor of "$\neg$" and "$\wedge$".

   Exercise 2:

   b1) Prove that you can derive $\neg(P \wedge \neg Q)$ from $P \rightarrow Q$.
   b2) Prove that you can derive $P \rightarrow Q$ from $\neg(P \wedge \neg Q)$.

   Recall that on (the washed out) problem set 7, we proved that $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$. This tells us that after we eliminate "$\rightarrow$" from our language, we can also eliminate either "$\vee$" or "$\wedge$". (Make sure you understand why this means we can "in principle" make do with just "$\vee$" and "$\neg$" or just "$\wedge$" and "$\neg$" as our only connectives.)

3. **Exercise 3 i):** Translate the following sentences into logically equivalent sentences containing $\vee$ and $\neg$ as the only connectives. (Just find and write the translation. You don't actually have to write out the proofs demonstrating that the sentences are logically equivalent.)

   a) $P \to (Q \to \neg R)$
   b) $(P \wedge Q) \vee (P \wedge R)$

   Exercise 3ii)
   Now draw truth-tables that establish that the given sentence and your translation are equivalent.

4. In fact, you only need *one* connective to define all possible Boolean formulae. As we'll see in the next chapter, we can define one connectives that will work to define all connectives. In fact, there are two possible connectives of that type. Here are introduction and elimination rules for one of them, which we'll call "neither… nor …", and which we'll write "$\downarrow$":

$$\begin{array}{|l} P \downarrow Q \\ \\ \neg P \\ \\ \end{array} \qquad \begin{array}{|l} P \downarrow Q \\ \\ \neg Q \\ \\ \end{array} \qquad \begin{array}{|l} \neg P \\ \\ \neg Q \\ \\ Q \downarrow P \\ \\ \end{array}$$

Exercise 4)
Prove, in the system that results from adding the symbol and rules for “neither… nor …” to boolean propositional logic, that $P \downarrow P$ is logically equivalent to $\neg P$ and $(P \downarrow Q) \downarrow (P \downarrow Q)$ is logically equivalent to $P \vee Q$.

To give you the idea of what I’m looking for in Exercise 4, I’ll prove one direction of each equivalence. You will need to prove the other direction:

1	$P \downarrow P$	hyp
2	$\neg P$	nn elim. 1

1	$(P \downarrow Q) \downarrow (P \downarrow Q)$	hyp
2	$\neg(P \downarrow Q)$	nn elim 1
3	$\quad \neg(P \vee Q)$	hyp
4	$\quad\quad \neg P$	hyp
5	$\quad\quad\quad \neg Q$	hyp
6	$\quad\quad\quad (P \downarrow Q)$	nn intro 4 5
7	$\quad\quad\quad \neg(P \downarrow Q)$	reit 2
8	$\quad\quad \neg\neg Q$	neg intro 5–7
9	$\quad\quad Q$	neg elim 8
10	$\quad\quad P \vee Q$	disj intro 9
11	$\quad\quad \neg(P \vee Q)$	reit 3
12	$\quad \neg\neg P$	neg intro 4–11
13	$\quad P$	neg elim 12
14	$\quad P \vee Q$	disj intro 13
15	$\quad \neg(P \vee Q)$	reit 3
16	$\neg\neg(P \vee Q)$	neg intro 3–15
17	$P \vee Q$	neg elim 16

5. **Exercise 5:**
   Given the rules of introduction and elimination for $\downarrow$, figure out what its truth-table looks like and write it down.

6. **Exercise 6i):**
   Prove using the rules of chapter IV that these two sentences are logically equivalent:

   $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$

   Exercise 6ii):
   Then construct a truth-table to show that these two sentences are truth-functionally equivalent: