

## Homework Assignment 11 Solutions

1. Chapter 8, Exercise 1 To get everything in on a single line, I'll replace "VotesFor" with "V"

(a) Legitimate vote:

$$\neg V(a, a) \wedge \neg V(b, b) \wedge \neg V(c, c)$$

(b) Legitimate but deadlocked:

$$\neg V(a, a) \wedge \neg V(b, b) \wedge \neg V(c, c) \wedge V(a, b) \wedge V(b, c) \wedge V(c, a) \wedge V(a, c) \wedge V(b, a) \wedge V(c, b) \text{ (The first clause } \neg V(a, a) \wedge \neg V(b, b) \wedge \neg V(c, c) \text{ is actually redundant, but it doesn't hurt to put it in.)}$$

(c) Legitimate and x is the winner:

$$(\neg V(a, a) \wedge \neg V(b, b) \wedge \neg V(c, c)) \wedge ((V(a, x) \wedge V(b, x)) \vee (V(b, x) \wedge V(c, x)) \vee (V(a, x) \wedge V(c, x)))$$

(d) Legitimate and x is a minority voter:

$$\begin{aligned} &(\neg V(a, a) \wedge \neg V(b, b) \wedge \neg V(c, c)) \\ &\wedge ((V(x, b) \vee V(x, c)) \wedge V(c, a) \wedge V(b, a)) \vee ((V(x, a) \vee V(x, c)) \wedge V(c, b) \wedge V(a, b)) \\ &\vee ((V(x, a) \vee V(x, b)) \wedge V(a, c) \wedge V(b, a)) \end{aligned}$$

(e) Unanimous:  $(V(a, a) \wedge V(b, a) \wedge V(c, a)) \vee (V(a, b) \wedge V(b, b) \wedge V(c, b)) \vee (V(a, c) \wedge V(b, c) \wedge V(c, c))$

(f) This is one of the rare cases where we use disjunction elimination with *three* constituents (Rule 92 in the text). We need to tinker a bit in the course of each of the sub-derivations, to ensure that we have the same contradiction in each case, so we can apply disj. elim.. Also, in this derivation, to fit everything on one line, I'll write  $V(x,x)$  as  $Vx$  when  $x$  is  $a$ ,  $b$ , or  $c$ .



Say that for a given  $n$  there is a sequence  $a_1, a_2, \dots, a_{n-1}$  of  $n - 1$  members of  $D$  such that for every  $i$ , such that  $1 < i < n$ ,  $R(a_{i-1}, a_i)$ .

There must be some  $a_n \in D$  such that  $R(a_{n-1}, a_n)$ . We can show that  $a_n \neq a_1$  since by repeated applications of transitivity as above we can show that  $R(a_1, a_{n-1})$ , so if we also had  $R(a_{n-1}, a_1)$  and applying transitivity we get  $R(a_1, a_1)$  which contradicts irreflexivity.

Similar reasoning shows that  $a_n \neq a_i$  for any  $i < n$ . So  $a_n$  is a further element of  $D$ , distinct from  $a_1, a_2, \dots, a_{n-1}$ .

Hence there must be at least  $n$  members of  $D$  for every  $n$ .

An example of an infinite domain with such a relation  $R$  is the natural numbers  $\mathbb{N}$  under the relation  $x < y$ .

3. Chapter 8, Exercise 12

1	$a = b$	hyp		
2	$c = b$	hyp		
3	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-top: 1px solid black; border-left: 1px solid black; padding-left: 10px;"><math>a = c</math></td> <td style="padding-left: 10px;">1,2 id. elim.</td> </tr> </table>	$a = c$	1,2 id. elim.	
$a = c$	1,2 id. elim.			

4. Chapter 8, Exercise 13

1	$P(a)$	hyp		
2	$\neg P(b)$	hyp		
3	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-top: 1px solid black; border-left: 1px solid black; padding-left: 10px;"><math>a = b</math></td> <td style="padding-left: 10px;">hyp.</td> </tr> </table>	$a = b$	hyp.	
$a = b$	hyp.			
4	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-top: 1px solid black; border-left: 1px solid black; padding-left: 10px;"><math>P(a)</math></td> <td style="padding-left: 10px;">R 1</td> </tr> </table>	$P(a)$	R 1	
$P(a)$	R 1			
5	$\neg P(a)$	id. elim. 2,3		
6	$\neg a = b$	neg. intro 3 - 5		

5. Chapter 8, Exercise 17

1	$\neg R(x, x)$	axiom
2	$(R(x, y) \wedge R(y, z)) \rightarrow R(x, z)$	axiom
3	$R(a, b)$	hyp.
4	$R(b, c)$	hyp.
5	$R(c, a)$	hyp.
6	$(R(a, y) \wedge R(y, z)) \rightarrow R(a, z)$	instantiation 2
7	$(R(a, b) \wedge R(b, z)) \rightarrow R(a, z)$	instantiation 6
8	$(R(a, b) \wedge R(b, c)) \rightarrow R(a, c)$	instantiation 7
9	$R(a, b) \wedge R(b, c)$	conj. intro. 3,4
10	$R(a, c)$	modus ponens 8,9
11	$(R(a, y) \wedge R(y, z)) \rightarrow R(a, z)$	instantiation 2
12	$(R(a, c) \wedge R(c, z)) \rightarrow R(a, z)$	instantiation 11
13	$(R(a, c) \wedge R(c, a)) \rightarrow R(a, a)$	instantiation 12
14	$R(a, c) \wedge R(c, a)$	conj. intro. 5, 10
15	$R(a, a)$	modus ponens 13,14
16	$\neg R(a, a)$	instantiation 1
17	$\neg R(c, a)$	neg. intro 6-16

6. Produce a fifth static tictactoe language: this one will have coordinates  $\{1, 2, 3\}$  for the columns, and  $\{1, 2, 3\}$  for the rows. (To describe the location of an X or an O you will need two variable places: one for the column coordinate, and one for the row coordinate.). In this language, give expressions that are true of all and only the situations in which a) There are three X's in a line b) There are three O's in a line. To see the increase in simplicity, write down the expressions for a) and b) in the first static tictactoe language from the earlier chapters.

Three X's:  $(X(1, x) \wedge X(2, x) \wedge X(3, x)) \vee (X(x, 1) \wedge X(x, 2) \wedge X(x, 3))$   
 $\vee (X(1, 1) \wedge X(2, 2) \wedge X(3, 3)) \vee (X(1, 3) \wedge X(2, 2) \wedge X(3, 1))$

Three O's is the same as the above, except with O in place of X.

I'll leave it to you to write down the comparable expressions in other languages.