Philosophy 303
Symbolic Logic
Winter, 2009

Homework Assignment 10 Solutions

1. Say that we add to our rules of inference the rule of Resolution:

j	$P \vee \neg Q$	
	$\vdots$	
k	$Q \vee R$	
	$\vdots$	
l	$P \vee R$	j, k *Resolution*

Prove that the new system is sound.

Answer:

We want to show that it is impossible for a use of resolution to take you from true premises to a false conclusion. Say that we have an instance of resolution, with truth - table given by the table below. The only lines on which both $P \vee \neg Q$ and $Q \vee R$ are true are indicated by the arrows. On all of those lines, $P \vee R$ is also true. So it is not possible to find an interpretation in which the premises of an application of resolution are true and the conclusion is false.

P	Q	R	P	∨	¬	Q	Q	∨	R)	P	∨	R
T	T	T	T	**T**	F	T	T	**T**	T	T	**T**	F ⇐
T	T	F	T	**T**	F	T	T	**T**	F	T	**T**	F ⇐
T	F	T	T	**T**	T	F	F	**T**	T	T	**T**	T ⇐
T	F	F	T	**T**	T	F	F	**F**	F	T	**T**	T
F	T	T	F	**F**	F	T	T	**T**	T	F	**T**	F
F	T	F	F	**F**	F	T	T	**T**	F	F	**F**	F
F	F	T	F	**T**	T	F	F	**T**	T	F	**T**	T ⇐
F	F	F	F	**T**	T	F	F	**F**	F	F	**F**	T

2. Say that we add to our rules of inference the rules of Axiom1 and Axiom2. Axiom1 says that on any line of a derivation, you may write down the axiom $(P \rightarrow (Q \rightarrow P))$, subject to all and only only the hypotheses that governed the previous line. Axiom2 says that on any line of a derivation, you may write down the axiom $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$, subject to all and only only the hypotheses that governed the previous line.

Prove that the new system is sound.

Answer:

This one is a bit unusual, because you don't need to be concerned about premises. Both of these sentences are true no matter what the interpretation is. So you don't need to worry that by writing down one of these axioms, you are introducing a false sentence into the derivation. For proof, we need only look at the truth tables.

P	Q	P	→	(Q	→	P)
T	T	T	**T**	T	T	T
T	F	T	**T**	F	T	T
F	T	F	**T**	T	F	F
F	F	F	**T**	F	T	F

P	Q	R	P	→	(Q	→	R)	→	(P	→	Q)	→	(P	→	R)
T	T	T	T	T	T	T	T	**T**	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F	**T**	T	T	T	F	T	F	F
T	F	T	T	T	F	T	T	**T**	T	F	F	T	T	T	T
T	F	F	T	T	F	T	F	**T**	T	F	F	T	T	F	F
F	T	T	F	T	T	T	T	**T**	F	T	T	T	F	T	T
F	T	F	F	T	T	F	F	**T**	F	T	T	T	F	T	F
F	F	T	F	T	F	T	T	**T**	F	T	F	T	F	T	T
F	F	F	F	T	F	T	F	**T**	F	T	F	T	F	T	F

3. Say that Γ is **V**-saturated, and A and B are formulas of **V**.
Show that:

$(\neg A \wedge \neg B) \in \Gamma$ if and only if $A \notin \Gamma$ and $B \notin \Gamma$

Answer:
This is an "if and only if" statement, so we need to prove both directions.
$\Rightarrow$
Say that $(\neg A \wedge \neg B) \in \Gamma$.
Assume also (*to derive a contradiction*) that $A \in \Gamma$ or $B \in \Gamma$.
First consider the possibility that $A \in \Gamma$.
Then we can get a very simple derivation of a contradiction from members of Γ:

1	$(\neg A \wedge \neg B)$	hyp
2	A	hyp
3	$\neg A$	conj elim 1

Now consider the possibility that $B \in \Gamma$.
Then we can get essentially the same derivation of a contradiction from members of Γ:

1	$(\neg A \wedge \neg B)$	hyp
2	B	hyp
3	$\neg B$	conj elim 1

But this contradicts the assumption that Γ is **V**-saturated, since (by definition) you cannot derive contradictions from **V**-saturated sets.

This shows that the assumption $A \in \Gamma$ or $B \in \Gamma$ is false.
Remembering our logical equivalences, the negation of "$A \in \Gamma$ or $B \in \Gamma$" is "$A \notin \Gamma$ *and* $B \notin \Gamma$", which is what we want to prove.

$\Leftarrow$

Say that $A \notin \Gamma$ and $B \notin \Gamma$

Since Γ is **V**-saturated, we can conclude $\neg A \in \Gamma$ and $\neg B \in \Gamma$.

This gives us an easy derivation to show that $\Gamma \vdash (\neg A \wedge \neg B)$:

1	$\neg A$	hyp
2	$\neg B$	hyp
3	$(\neg A \wedge \neg B)$	conj intro 1, 2

Metatheorem 27 tells us that if Γ is **V**-saturated and $\Gamma \vdash \phi$, then $\phi \in \Gamma$. So, in particular, we know that $(\neg A \wedge \neg B) \in \Gamma$, which is what we needed to prove.

This proves the $\Leftarrow$ direction, and both directions together prove the theorem.

$(\neg A \rightarrow B) \in \Gamma$ if and only if $A \in \Gamma$ or $B \in \Gamma$

Answer:
This is an “if and only if” statement, so we need to prove both directions.

$\Rightarrow$

Say that $(\neg A \rightarrow B) \in \Gamma$.
Assume also (*to derive a contradiction*) that $A \notin \Gamma$ and $B \notin \Gamma$.

Since Γ is **V**-saturated, we can conclude $\neg A \in \Gamma$ and $\neg B \in \Gamma$.

Then we can get a very simple derivation of a contradiction from members of Γ:

1	$(\neg A \rightarrow B)$	hyp
2	$\neg A$	hyp
3	$\neg B$	hyp
4	B	Modus Ponens 1,2
5	$\neg B$	R 3

But this contradicts the assumption that Γ is **V**-saturated, since (by definition) you cannot derive contradictions from **V**-saturated sets.

This shows that the assumption $A \notin \Gamma$ and $B \notin \Gamma$ is false.
Remembering our logical equivalences, the negation of “$A \notin \Gamma$ or $B \notin \Gamma$” is “$A \in \Gamma$ *or* $B \in \Gamma$”, which is what we want to prove.

$\Leftarrow$

Say that $A \in \Gamma$ or $B \in \Gamma$

Consider first the possibility that $A \in \Gamma$.

We can show $\Gamma \vdash (\neg A \rightarrow B)$:

1	A	hyp
2	$\neg A$	hyp
3	$\neg B)$	conj intro 1, 2
4	A	R 1
5	$\neg A$	R 2
6	$\neg\neg B$	neg. intro. 3 - 5
7	B	neg. elim. 6
8	$\neg A \rightarrow B$	Cond. intro. 2 - 7

Metatheorem 27 tells us that if Γ is **V**-saturated and $\Gamma \vdash \phi$, then $\phi \in \Gamma$. So, in particular, we know that $\neg A \rightarrow B \in \Gamma$, which is what we needed to prove.

Consider now the possibility that $B \in \Gamma$.

We can show $\Gamma \vdash (\neg A \rightarrow B)$:

1	B	hyp
2	$\neg A$	hyp
3	B	R 1
4	$\neg A \rightarrow B$	Cond. intro. 2 - 3

Metatheorem 27 tells us that if Γ is **V**-saturated and $\Gamma \vdash \phi$, then $\phi \in \Gamma$. So, in particular, we know that $\neg A \rightarrow B \in \Gamma$, which is what we needed to prove.

Taking both possibilities together proves the $\Leftarrow$ direction, and both directions together prove the theorem.