Philosophy 303
Symbolic Logic
Winter, 2009

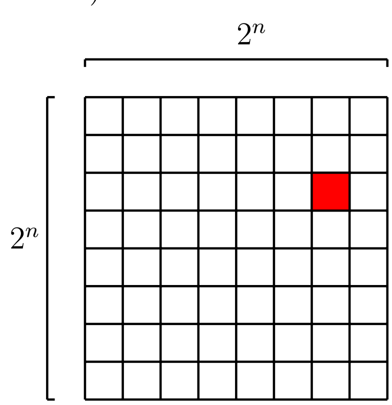
Homework Assignment 3

Due Friday Jan. 30, 2009 by 3 PM

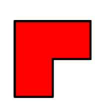
The problems draw from different parts of the book, but some sections are especially important. For these problems, pay special attention to pages 61 - 76 and the posted recursion notes;

1. Call a string over $\{a, b, c\}$ an "a-palindrome" if it is a palindrome that has "a" as the middle letter (if it has an odd number of symbols) and "aa" as a middle string (if it has an even number of symbols). Give a recursive definition of "a - palindrome", and prove by induction that every a-palindrome has an even number of "b"'s and "c"'s. (This one is meant to be quite straightforward, and just to give you practice in setting up recursive definitions and giving arguments in inductive form. Don't make it more complicated than it has to be.)

2. Prove that any square chessboard that is 2^n squares wide and 2^n squares high, with one square removed, can be completely covered (with no overlap) by a collection of tiles, where each tile is three squares in an L-shape. In pictures:
   Say you have a 2^n x 2^n chessboard, with one square removed (removed square marked in red).

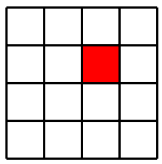
   

   (Note: the chessboard in the diagram is 8 x 8, but it is meant to represent an arbitrary 2^n x 2^n board.)

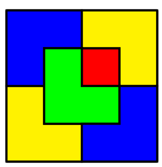
   The white squares can all be covered (with no overlap) by tiles in this shape:  .

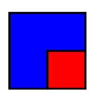
Remarks about this problem:

a) To give you an idea of what we are trying to prove, here is an example of a 4 x 4 chessboard (i.e. 2^2 x 2^2) with one square removed (in red).

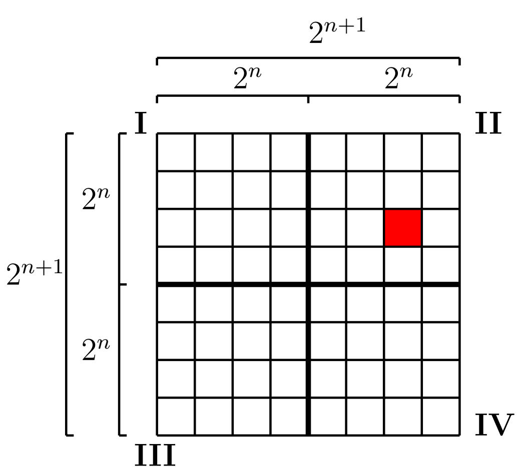


The above chessboard can be L-tiled this way. (The L-shaped tiles are displayed in different colors, for easier viewing.) You want to show that this kind of a tiling will be possible for any 2^n x 2^n board with one square removed.



b) You will have to proceed by induction. The base case, for a 2 x 2 board, is automatic. Whatever square is chosen, there will be an L-shape left over, as in this case (chosen square in red, remainder in blue): 

c) The point of the inductive proof is to reduce the problem of tiling a 2^{n+1} x 2^{n+1} board to some question or family of questions about tiling 2^n x 2^n boards, since 2^n x 2^n boards are covered by the induction hypothesis. In a picture:



By dividing the 2^{n+1} x 2^{n+1} board into 4 quadrants, each of size 2^n x 2^n we get a situation where the inductive hypothesis can help us. The red square will be in one of the quadrants - in the diagram above, it is in the upper right hand quadrant, labelled II. By the induction hypothesis we know that that quadrant II can be covered in L-tiles, leaving the red square uncovered. **But** this leaves us with one more problem to solve before we can conclude that every 2^{n+1} x 2^{n+1} board with one square missing can be covered in L-tiles: How can we show that the rest of the board (quadrants I, III, and IV in the above diagram) can be tiled with L-tiles? The induction hypothesis just addresses chessboards with one square missing, not chessboards with all the squares in place. Here we need a bit of inspiration to figure out how to produce a situation where we can use the induction hypothesis. Once that is done, the problem falls into place.

Hint: We can get a situation to which the inductive hypothesis applies to each of the quadrants without the red square by placing one L-tile on the part of the board consisting of the quadrants without the red square. (In our diagram: by placing one L-tile on the part of the board consisting of quadrants I, III, and IV, we can get a situation where the induction hypothesis applies to quadrants I, III, and IV.)

3. Here is the inductive definition of n! (read "n factorial"):
   $1! = 1$
   $(n+1)! = (n+1) \times n!$

   That is, $n! = \underbrace{(n \times (n-1) \times \ldots 3 \times 2 \times 1)}_{n\ times}$

   Prove by induction: For every n greater than 3, $2^n < n! < n^n$.

4. Say that in addition to the nickel, we had a 3 cent coin. Prove by induction that every amount of money above 7 cents could be made up of some collection of 3 cent coins and 5 cent coins.