

## Worked exercises for problem sets 8 and 9

1. A *derived rule* is a rule whose correctness is proven by showing that it works as an abbreviation of other, basic rules. The idea is that when you apply a derived rule, you could have written the longer derivation in terms of basic rules instead, but the derived rule is like a “macro” that saves you time and space.

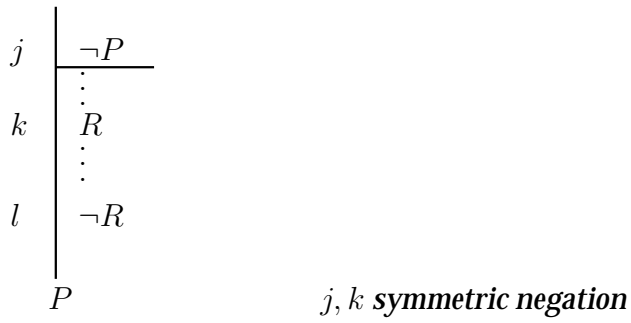
Prove that *shortcut* is a correct derived rule, where *shortcut* is the rule:

$j$	$P \rightarrow Q$	
	$\vdots$	
$k$	$Q \rightarrow R$	
	$\vdots$	
$l$	$\neg R \rightarrow \neg P$	$j, k$ <i>shortcut</i>

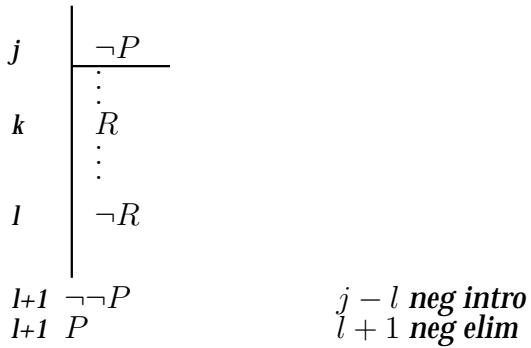
**Solution:**

$j$	$P \rightarrow Q$	
	$\vdots$	
$k$	$Q \rightarrow R$	
	$\vdots$	
$l$	$\neg R$	<i>hyp</i>
$l+1$	$P$	<i>hyp</i>
$l+2$	$Q$	$j, l+1$ , <i>modus ponens</i>
$l+3$	$R$	$k, l+2$ , <i>modus ponens</i>
$l+4$	$\neg R$	$l$ , <i>reit</i>
$l+5$	$\neg P$	$l-l+4$ , <i>neg intro</i>
$l+5$	$\neg R \rightarrow \neg P$	$l-l+5$ , <i>cond intro</i>

2. Prove that *symmetric negation* is a correct derived rule, where *symmetric negation* is the rule:



**Solution:**



3. Chapter 5

exercise 7 a)

P	(P → ¬ P)	→	P
T	F	T	F
F	T	F	T

Not valid, Satisfiable.

Example:  $\neg P \rightarrow \neg\neg P$

P	(¬ P → ¬ ¬ P)
T	F
F	T

Not valid, Satisfiable.

7 g)

P	$\neg$	P
T	<b>F</b>	<b>T</b>
F	<b>T</b>	<b>F</b>

Not Valid, Satisfiable

7 m)

P	Q	(P	$\vee$	Q)	$\rightarrow$	P
T	T	T	T	T	<b>T</b>	<b>T</b>
T	F	T	T	F	<b>T</b>	<b>T</b>
F	T	F	T	T	<b>F</b>	<b>F</b>
F	T	F	F	F	<b>T</b>	<b>T</b>

Not Valid, Satisfiable

7 u)

P	Q	P	$\wedge$	Q	$\wedge$	$\neg P$
T	T	T	F	T	F	F
T	F	T	F	F	F	F
F	T	F	F	T	F	T
F	T	F	F	F	F	T

Not valid, Unsatisfiable.

4. Chapter 5 exercise 10

A	B	A	$\rightarrow$	(B	$\rightarrow$	A)
T	T	T	T	T	T	T
T	F	T	T	F	T	T
F	T	F	T	T	T	F
F	T	F	T	F	T	F

A	B	C	A	$\rightarrow$	(B	$\rightarrow$	C)	$\rightarrow$	A	$\rightarrow$	B	$\rightarrow$	A	$\rightarrow$	C)
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F	T	T	T	T	F	T	F	F
T	F	T	T	T	F	T	T	T	T	F	F	T	T	T	T
T	F	F	T	T	F	T	F	T	T	F	F	T	T	F	F
F	T	T	F	T	T	T	T	T	F	T	T	T	F	T	T
F	T	F	F	T	T	F	F	T	F	T	T	T	F	T	F
F	F	T	F	T	F	T	T	T	F	T	F	T	F	T	T
F	F	F	F	T	F	T	F	T	F	T	F	T	F	T	F

A	B	( $\neg$	B	$\rightarrow$	$\neg$	A)	$\rightarrow$	(A	$\rightarrow$	B)
T	T	F	T	T	F	T	T	T	T	T
T	F	T	F	F	F	T	T	T	F	F
F	T	F	T	T	T	F	T	F	T	T
F	T	T	F	T	T	F	T	F	T	T

A	B	(A	$\rightarrow$	B)	$\rightarrow$	((A	$\rightarrow$	$\neg$	B)	$\rightarrow$	$\neg$	A)
T	T	T	T	T	T	F	F	T	T	F	T	
T	F	T	F	F	T	T	T	F	F	F	T	
F	T	F	T	T	T	F	T	F	T	T	F	
F	T	F	T	F	T	F	T	T	F	T	F	

	A	B	A	$\rightarrow$	((A	$\rightarrow$	B)	$\rightarrow$	B
(e)	T	T	T	T	T	T	T	T	T
	T	F	T	T	T	F	F	T	F
	F	T	F	T	F	T	T	T	T
	F	T	F	T	F	T	F	F	F

	A	B	$\neg$	A	$\rightarrow$	(A	$\rightarrow$	B)
(f)	T	T	F	T	T	T	T	T
	T	F	F	T	T	T	F	F
	F	T	T	F	T	F	T	T
	F	T	T	F	T	F	T	F

5. From chapter 5, page 38, prove Metatheorems 23, 24, 26 and 31.

**Metatheorem 23:** If  $\Gamma \Vdash (A \rightarrow B)$ , then  $\Gamma \cup \{A\} \Vdash B$

**Proof:**

Say we have an interpretation  $\hat{I}$  that makes every sentence in  $\Gamma \cup \{A\}$  true. Then  $\hat{I}$  makes every sentence in  $\Gamma$  true, so it makes  $A \rightarrow B$  true. Also,  $\hat{I}$  makes A true. This restricts us to the line of the truth table in boldface:

A	B	$A \rightarrow B$
<b>T</b>	<b>T</b>	<b>T</b>
T	F	F
F	T	T
F	F	T

Since B is true on the given line of the truth table,  $\hat{I}$  makes B true.

Hence  $\Gamma \cup \{A\} \Vdash B$

**Metatheorem 24:** If  $\Gamma \cup \{A\} \Vdash B$  then  $\Gamma \Vdash (A \rightarrow B)$ .

Say we have an interpretation  $\hat{I}$  that makes every sentence in  $\Gamma$  true. If  $\hat{I}$  makes A false then  $\hat{I}$  makes  $A \rightarrow B$  true, since an  $\rightarrow$  sentence with a false antecedent is true. If  $\hat{I}$  makes A true, then  $\hat{I}$  makes every sentence in  $\Gamma \cup \{A\}$  true, so  $\hat{I}$  makes B true. Hence we are restricted to the boldface line of the truth table:

A	B	$A \rightarrow B$
<b>T</b>	<b>T</b>	<b>T</b>
T	F	F
F	T	T
F	F	T

Since  $A \rightarrow B$  is true on the given line of the truth table,  $\hat{I}$  makes  $A \rightarrow B$  true.

Hence  $\Gamma \Vdash (A \rightarrow B)$ .

**Metatheorem 26.** The point here is that no interpretation can make both  $B$  and  $\neg B$  true, so no interpretation can make every sentence in  $\Gamma \cup \{A\}$  true. So if any interpretation makes every sentence in  $\Gamma$  true, it cannot make  $A$  true, so it must make  $\neg A$  true.

**Metatheorem 31** Any interpretation  $\hat{I}$  making every sentence in  $\Gamma \cup \{A \vee B\}$  true must make either  $A$  true or  $B$  true by the truth-table for  $\vee$ . Hence  $\hat{I}$  makes either every sentence in  $\Gamma \cup \{A\}$  true or every sentence in  $\Gamma \cup \{B\}$  true. Either way,  $\hat{I}$  must make  $C$  true, since  $\Gamma \cup \{A\} \Vdash C$  and  $\Gamma \cup \{B\} \Vdash C$

6. Chapter 5 exercise 16 (all four parts)

(a) If  $\Gamma$  is simultaneously satisfiable and  $\Delta$  is simultaneously satisfiable is  $\Gamma \cup \Delta$  simultaneously satisfiable?

No: Let  $\Gamma = \{P\}$  and  $\Delta = \{\neg P\}$

(b) If  $\Gamma$  is simultaneously satisfiable and  $\Delta$  is simultaneously satisfiable is  $\Gamma \cap \Delta$  simultaneously satisfiable?

Yes.  $\Gamma$  is simultaneously satisfiable so there is an interpretation  $\hat{I}$  making every sentence in  $\Gamma$  true. This interpretation must also make every sentence in  $\Gamma \cap \Delta$  true, since  $\Gamma \cap \Delta \subseteq \Gamma$ . (You don't even have to consider  $\Delta$ ).

(c) If  $\Gamma$  is simultaneously satisfiable and  $\Delta$  is simultaneously satisfiable is  $\{(A \vee B) \mid A \in \Gamma \text{ and } B \in \Delta\}$  simultaneously satisfiable?

Yes. Once again, you don't need to consider  $\Delta$  at all! Any interpretation making every sentence in  $\Gamma$  true must also make every sentence in  $\{(A \vee B) \mid A \in \Gamma \text{ and } B \in \Delta\}$  true, by the truth-table for  $\vee$ .

(d) If  $\Gamma$  is simultaneously satisfiable is  $\{\neg A \mid A \in \Gamma\}$  simultaneously satisfiable?

No. Let  $\Gamma = \{P \vee \neg P\}$