Philosophy 303
Symbolic Logic
Fall, 2009

Worked exercises for problem sets 8 and 9

1. A *derived rule* is a rule whose correctness is proven by showing that it works as an abbreviation of other, basic rules. The idea is that when you apply a derived rule, you could have written the longer derivation in terms of basic rules instead, but the derived rule is like a "macro" that saves you time and space.

   Prove that *shortcut* is a correct derived rule, where *shortcut* is the rule:

j	$P \to Q$	
	$\vdots$	
k	$Q \to R$	
	$\vdots$	
l	$\neg R \to \neg P$	j, k *shortcut*

   Solution:

j	$P \to Q$	
	$\vdots$	
k	$Q \to R$	
	$\vdots$	
l	│ $\neg R$	*hyp*
$l+1$	│ │ P	hyp
$l+2$	│ │ Q	$j, l+1,$ *modus ponens*
$l+3$	│ │ R	$k, l+2,$ *modus ponens*
$l+4$	│ │ $\neg R$	$l,$ *reit*
$l+5$	│ $\neg P$	$l - l+4,$ *neg intro*
$l+5$	$\neg R \to \neg P$	$l - l+5,$ *cond intro*

2. Prove that *symmetric negation* is a correct derived rule, where *symmetric negation* is the rule:

j	$\neg P$
	⋮
k	R
	⋮
l	$\neg R$

P j, k *symmetric negation*

Solution:

j	$\neg P$
	⋮
k	R
	⋮
l	$\neg R$

l+1 $\neg\neg P$ $j - l$ *neg intro*
l+1 P $l + 1$ *neg elim*

3. Chapter 5

exercise 7 a)

P	(P	→	¬	P)	→	P)
T	T	F	F	T	**T**	T
F	F	T	T	F	**F**	F

Not valid, Satisfiable.

Example: $\neg P \to \neg\neg P$

P	(¬	P	→	¬	¬	P)
T	F	T	**T**	T	F	T
F	T	F	**F**	F	T	F

Not valid, Satisfiable.

7 g)

P	¬	P
T	**F**	T
F	**T**	F

Not Valid, Satisfiable

7 m)

P	Q	(P	∨	Q)	→	P
T	T	T	T	T	**T**	T
T	F	T	T	F	**T**	T
F	T	F	T	T	**F**	F
F	T	F	F	F	**T**	T

Not Valid, Satisfiable

7 u)

P	Q	P	∧	Q	∧	¬P
T	T	T	F	T	**F**	F
T	F	T	F	F	**F**	F
F	T	F	F	T	**F**	T
F	T	F	F	F	**F**	T

Not valid, Unsatisfiable.

4. Chapter 5 exercise 10

(a)

A	B	A	→	(B	→	A)
T	T	T	**T**	T	T	T
T	F	T	**T**	F	T	T
F	T	F	**T**	T	T	F
F	T	F	**T**	F	T	F

(b)

A	B	C	A	→	(B	→	C)	→	A	→	B	→	A	→	C)
T	T	T	T	T	T	T	T	**T**	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F	**T**	T	T	T	F	T	F	F
T	F	T	T	T	F	T	T	**T**	T	F	F	T	T	T	T
T	F	F	T	T	F	T	F	**T**	T	F	F	T	T	F	F
F	T	T	F	T	T	T	T	**T**	F	T	T	T	F	T	T
F	T	F	F	T	T	F	F	**T**	F	T	T	T	F	T	F
F	F	T	F	T	F	T	T	**T**	F	T	F	T	F	T	T
F	F	F	F	T	F	T	F	**T**	F	T	F	T	F	T	F

(c)

A	B	(¬	B	→	¬	A)	→	(A	→	B)
T	T	F	T	T	F	T	**T**	T	T	T
T	F	T	F	F	F	T	**T**	T	F	F
F	T	F	T	T	T	F	**T**	F	T	T
F	T	T	F	T	T	F	**T**	F	T	T

(d)

A	B	(A	→	B)	→	((A	→	¬	B)	→	¬	A)
T	T	T	T	T	**T**	T	F	F	T	T	F	T
T	F	T	F	F	**T**	T	T	T	F	F	F	T
F	T	F	T	T	**T**	F	T	F	T	T	T	F
F	T	F	T	F	**T**	F	T	T	F	T	T	F

(e)

A	B	A	→	((A	→	B)	→	B
T	T	T	**T**	T	T	T	T	T
T	F	T	**T**	T	F	F	T	F
F	T	F	**T**	F	T	T	T	T
F	T	F	**T**	F	T	F	F	F

(f)

A	B	¬	A	→	(A	→	B)
T	T	F	T	**T**	T	T	T
T	F	F	T	**T**	T	F	F
F	T	T	F	**T**	F	T	T
F	T	T	F	**T**	F	T	F

5. From chapter 5, page 38, prove Metatheorems 23, 24, 26 and 31.

Metatheorem 23: If $\Gamma \Vdash (A \rightarrow B)$, then $\Gamma \cup \{A\} \Vdash B$

Proof:

Say we have an interpretation Î that makes every sentence in $\Gamma \cup \{A\}$ true. Then Î makes every sentence in Γ true, so it makes $A \rightarrow B$ true. Also, Î makes A true. This restricts us to the line of the truth table in boldface:

A	B	$A \rightarrow B$
T	**T**	**T**
T	F	F
F	T	T
F	F	T

Since B is true on the given line of the truth table, Î makes B true.

Hence $\Gamma \cup \{A\} \Vdash B$

Metatheorem 24: If $\Gamma \cup \{A\} \Vdash B$ then $\Gamma \Vdash (A \rightarrow B)$.

Say we have an interpretation Î that makes every sentence in Γ true. If Î makes A false then Î makes $A \rightarrow B$ true, since an $\rightarrow$ sentence with a false antecedent is true. If Î makes A true, then Î makes every sentence in $\Gamma \cup \{A\}$ true, so I makes B true. Hence we are restricted to the boldface line of the truth table:

A	B	$A \rightarrow B$
T	**T**	**T**
T	F	F
F	T	T
F	F	T

Since $A \rightarrow B$ is true on the given line of the truth table, Î makes $A \rightarrow B$ true.

Hence $\Gamma \Vdash (A \rightarrow B)$.

Metatheorem 26. The point here is that no interpretation can make both B and $\neg B$ true, so no interpretation can make every sentence in $\Gamma \cup \{A\}$ true. So if any interpretation makes every sentence in Γ true, it cannot make A true, so it must make $\neg A$ true.

Metatheorem 31 Any interpretation Î making every sentence in $\Gamma \cup \{A \vee B\}$ true must make either A true or B true by the truth-table for $\vee$. Hence Î makes either every sentence in $\Gamma \cup \{A\}$ true or every sentence in $\Gamma \cup \{B\}$ true. Either way, Î must make C true, since $\Gamma \cup \{A\} \Vdash C$ and $\Gamma \cup \{B\} \Vdash C$

6. Chapter 5 exercise 16 (all four parts)

   (a) If Γ is simultaneously satisfiable and Δ is simultaneously satisfiable is $\Gamma \cup \Delta$ simultaneously satisfiable?

   No: Let $\Gamma = \{P\}$ and $\Delta = \{\neg P\}$

   (b) If Γ is simultaneously satisfiable and Δ is simultaneously satisfiable is $\Gamma \cap \Delta$ simultaneously satisfiable?

   Yes. Γ is simultaneously satisfiable so there is an interpretation Î making every sentence in Γ true. This interpretation must also make every sentence in $\Gamma \cap \Delta$ true, since $\Gamma \cap \Delta \subseteq \Gamma$. (You don't even have to consider Δ).

   (c) If Γ is simultaneously satisfiable and Δ is simultaneously satisfiable is $\{(A \vee B) | A \in \Gamma \text{ and } B \in \Delta\}$ simultaneously satisfiable?

   Yes. Once again, you don't need to consider Δ at all! Any interpretation making every sentence in Γ true must also make every sentence in $\{(A \vee B) | A \in \Gamma \text{ and } B \in \Delta\}$ true, by the truth-table for $\vee$.

   (d) If Γ is simultaneously satisfiable is $\{\neg A | A \in \Gamma\}$ simultaneously satisfiable?

   No. Let $\Gamma = \{P \vee \neg P\}$