

Worked Examples for Problem Sets 6 and 7)

1. (Chapter 4, Exercise 5, Part (a).)

1	$(R \rightarrow Q)$	hyp		
2	$(P \rightarrow R)$	hyp		
3	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">P</td> <td>hyp</td> </tr> </table>	P	hyp	
P	hyp			
4	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">$(P \rightarrow R)$</td> <td>2, reit</td> </tr> </table>	$(P \rightarrow R)$	2, reit	
$(P \rightarrow R)$	2, reit			
5	R	3,4, m p		
6	$(R \rightarrow Q)$	1, reit		
7	Q	5,6, m p		
8	$(P \rightarrow Q)$	3-7, cond int		

2. (Chapter 4, Exercise 5, Part (c).)

1	Q	hyp				
2	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">P</td> <td>hyp</td> </tr> </table>	P	hyp			
P	hyp					
3	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;"> <table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">P</td> <td>hyp</td> </tr> </table> </td> <td></td> </tr> </table>	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">P</td> <td>hyp</td> </tr> </table>	P	hyp		
<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">P</td> <td>hyp</td> </tr> </table>	P	hyp				
P	hyp					
4	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">P</td> <td>Reit 3</td> </tr> </table>	P	Reit 3			
P	Reit 3					
5	$P \rightarrow P$	cond int 3-4				
6	$P \rightarrow (P \rightarrow P)$	cond int 2-5				

3. (Chapter 4, Exercise 5, part d))

1	$Q \rightarrow R$	hyp		
2	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">P</td> <td>hyp</td> </tr> </table>	P	hyp	
P	hyp			
3	<table style="border-collapse: collapse; margin-left: 5px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">P</td> <td>2 reit</td> </tr> </table>	P	2 reit	
P	2 reit			
4	$P \rightarrow P$	Cond int 2-3		

4. (Chapter 4, Exercise 5, part h))

1	$(Q \rightarrow P) \rightarrow R$	hyp		
2	P	hyp		
3	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">Q</td> <td></td> </tr> </table>	Q		hyp
Q				
4	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">P</td> <td></td> </tr> </table>	P		reit 2
P				
5	$Q \rightarrow P$	cond int 3-4		
6	R	mp 1 5		

5. (Chapter 4, Exercise 7, Part (a).)

1	$(P \vee Q)$	hyp		
2	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">P</td> <td></td> </tr> </table>	P		hyp
P				
3	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$(Q \vee P)$</td> <td></td> </tr> </table>	$(Q \vee P)$		2, dis int
$(Q \vee P)$				
4	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">Q</td> <td></td> </tr> </table>	Q		hyp
Q				
5	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$(Q \vee P)$</td> <td></td> </tr> </table>	$(Q \vee P)$		4, dis int
$(Q \vee P)$				
6	$(Q \vee P)$	1, 2-3, 4-5, dis elim		

6. (Chapter 4, Exercise 7, Part (c).)

1	$((P \vee Q) \wedge R)$	hyp		
2	$(P \vee Q)$	1, conj elim		
3	R	1, conj elim		
4	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">P</td> <td></td> </tr> </table>	P		hyp
P				
5	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">R</td> <td></td> </tr> </table>	R		3, reit
R				
6	$(P \wedge R)$	4,5, conj int		
7	$((P \wedge R) \vee Q)$	6, dis int		
8	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">Q</td> <td></td> </tr> </table>	Q		hyp
Q				
9	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$((P \wedge R) \vee Q)$</td> <td></td> </tr> </table>	$((P \wedge R) \vee Q)$		8, dis int
$((P \wedge R) \vee Q)$				
10	$((P \wedge R) \vee Q)$	4-7,8-9, dis elim		

7. (Chapter 4, Exercise 7, part e))

1	$(P \wedge Q) \rightarrow R$	hyp
2	P	hyp
3	Q	hyp
4	$P \wedge Q$	conj intro 2,3
5	R	mp 1-4
6	$Q \rightarrow R$	cond intro 3-5
7	$P \rightarrow (Q \rightarrow R)$	cond intro 2-7

8. (Chapter 4, Exercise 7, part f))

1	R	hyp
2	$\neg R$	hyp
3	R	reit 1
4	$\neg R$	reit 2
5	$\neg\neg R$	neg intro 2-4

9. (Chapter 4, Exercise 7, part g))

1	$\neg P$	hyp
2	P	hyp
3	$\neg(Q \vee R)$	hyp
4	P	reit 2
5	$\neg P$	reit 1
6	$\neg\neg(Q \vee R)$	neg intro 3-5
7	$(Q \vee R)$	neg elim 6
8	$P \rightarrow (Q \vee R)$	cond intro 2-7

10. (Chapter 4, Exercise 7, part h))

1	$P \rightarrow \neg P$	hyp			
2	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="padding-right: 10px;">2</td> <td style="border-left: 1px solid black; padding-left: 10px;">P</td> <td style="padding-left: 10px;">hyp</td> </tr> </table>	2	P	hyp	
2	P	hyp			
3	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="padding-right: 10px;">3</td> <td style="border-left: 1px solid black; padding-left: 10px;">P</td> <td style="padding-left: 10px;">reit 2</td> </tr> </table>	3	P	reit 2	
3	P	reit 2			
4	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="padding-right: 10px;">4</td> <td style="border-left: 1px solid black; padding-left: 10px;">$\neg P$</td> <td style="padding-left: 10px;">mp 1 2</td> </tr> </table>	4	$\neg P$	mp 1 2	
4	$\neg P$	mp 1 2			
5	$\neg P$	neg intro 2-4			

11. Give a categorical proof of $(P \vee \neg P)$

1	$\neg(P \vee \neg P)$	hyp			
2	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="padding-right: 10px;">2</td> <td style="border-left: 1px solid black; padding-left: 10px;">P</td> <td style="padding-left: 10px;">hyp</td> </tr> </table>	2	P	hyp	
2	P	hyp			
3	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="padding-right: 10px;">3</td> <td style="border-left: 1px solid black; padding-left: 10px;">$(P \vee \neg P)$</td> <td style="padding-left: 10px;">disj. intro 2</td> </tr> </table>	3	$(P \vee \neg P)$	disj. intro 2	
3	$(P \vee \neg P)$	disj. intro 2			
4	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="padding-right: 10px;">4</td> <td style="border-left: 1px solid black; padding-left: 10px;">$\neg(P \vee \neg P)$</td> <td style="padding-left: 10px;">Reit 1</td> </tr> </table>	4	$\neg(P \vee \neg P)$	Reit 1	
4	$\neg(P \vee \neg P)$	Reit 1			
5	$\neg P$	neg int 2-4			
6	$P \vee \neg P$	disj int 5			
7	$\neg(P \vee \neg P)$	reit 1			
8	$\neg\neg(P \vee \neg P)$	neg int 1-7			
9	$P \vee \neg P$	neg elim 8			

12. Prove Q from $(P \rightarrow Q)$ and $(\neg P \rightarrow Q)$

This isn't the cleanest proof, but it illustrates that once you have a proof of one theorem, you can reuse it, by pasting it into other proofs. For example, we can put the categorical proof of $P \vee \neg P$ given above to work within this derivation:

1	$(P \rightarrow Q)$	hyp
2	$(\neg P \rightarrow Q)$	hyp
3	$\neg(P \vee \neg P)$	hyp
4	$\begin{array}{ l} \hline P \\ \hline \end{array}$	hyp
5	$\begin{array}{ l} \hline (P \vee \neg P) \\ \hline \end{array}$	disj. intro 2
6	$\begin{array}{ l} \hline \neg(P \vee \neg P) \\ \hline \end{array}$	Reit 1
7	P	neg int 2-4
8	$P \vee \neg P$	disj int 5
9	$\neg(P \vee \neg P)$	reit 1
10	$P \vee \neg P$	neg int 1-7
11	$\begin{array}{ l} \hline P \\ \hline \end{array}$	hyp
12	$\begin{array}{ l} \hline Q \\ \hline \end{array}$	modus ponens 1,12
13	$\begin{array}{ l} \hline \neg P \\ \hline \end{array}$	hyp
14	$\begin{array}{ l} \hline Q \\ \hline \end{array}$	modus ponens 2,14
15	Q	disj elim 8, 12-13,14-15

13. Derive $\neg A \wedge \neg B$ from $\neg(A \vee B)$.

1	$\neg(A \vee B)$	hyp
2	A	hyp
3	$A \vee B$	disj intro 2
4	$\neg(A \vee B)$	reit 1
5	$\neg A$	neg intro 2-4
6	B	hyp
7	$A \vee B$	disj intro 6
8	$\neg(A \vee B)$	reit 1
9	$\neg B$	neg intro 6-8
10	$\neg A \wedge \neg B$	conj intro 5,9

14. Derive $\neg(A \wedge B)$ from $\neg A \vee \neg B$.

1	$\neg A \vee \neg B$	hyp
2	$\neg A$	hyp
3	$A \wedge B$	hyp
4	A	conj. elim. 3
5	$\neg A$	reit 2
6	$\neg(A \wedge B)$	neg intro 3-5
7	$\neg B$	hyp
8	$A \wedge B$	hyp
9	B	conj. elim. 8
10	$\neg B$	reit 7
11	$\neg(A \wedge B)$	neg intro 8-10
12	$\neg(A \wedge B)$	disj elim 1 2-6 7-11

15. Derive $\neg(A \vee B)$ from $\neg A \wedge \neg B$. The trick here is to note that you can get a contradiction from A and you can get a contradiction from B, but it is a different contradiction each time. So you need to get the same contradiction to apply disjunction elimination usefully.

1	$\neg A \wedge \neg B$	hyp
2	$A \vee B$	hyp
3	A	hyp
4	A	reit 3
5	$\neg A$	conj elim 1
6	$A \wedge \neg A$	conj intro 4,5
7	B	hyp
8	$\neg(A \wedge \neg A)$	hyp
9	B	reit 7
10	$\neg B$	conj elim 1
11	$\neg\neg(A \wedge \neg A)$	neg intro 8-10
12	$A \wedge \neg A$	neg elim 11
13	$A \wedge \neg A$	disj intro 2 3-6 7-12
14	A	conj elim 13
15	$\neg A$	conj elim 13
16	$\neg(A \vee B)$	neg intro 2-13

16. Derive $\neg A \vee \neg B$ from $\neg(A \wedge B)$.

1	$\neg(A \wedge B)$	hyp
2	$\neg(\neg A \vee \neg B)$	hyp
3	$\neg A$	hyp
4	$\neg A \vee \neg B$	disj intro 3
5	$\neg(\neg A \vee \neg B)$	reit 1
6	$\neg\neg A$	neg intro 5
7	A	neg elim 6
8	$\neg B$	hyp
9	$\neg A \vee \neg B$	disj intro 8
10	$\neg(\neg A \vee \neg B)$	reit 1
11	$\neg\neg B$	neg intro 10
12	B	neg elim 11
13	$A \wedge B$	conj intro 7, 12
14	$\neg(A \wedge B)$	reit 1
15	$\neg\neg(\neg A \vee \neg B)$	neg intro 2-14
16	$\neg A \vee \neg B$	neg elim 15