

## Some worked examples for problem set 2

### Chapter 2 Exercise 30:

1.  $\epsilon x = x$ . Defining equation for  $\epsilon$ .
2.  $x\epsilon x = xx$ . From 1, by concatenating  $x$  with each side.
3.  $x\epsilon = x$ . From 2, by association (regrouping  $\epsilon$  with the  $x$  to its right), and left decomposability.

### Chapter 2, Exercise 31:

Given the equation  $x\epsilon = x$  rather than  $\epsilon x = x$ , prove the empty string is unique.

The point of this problem is to give you practice spelling out things you would normally take to be obvious. It would seem obvious, given the definition of the empty string, that you should be able to stick it on *either side* of  $x$  and just get  $x$  as a result.

Observation: To prove uniqueness of "blahs", assume that you have *two* "blahs", and prove that these two must in fact be the same thing.

1. Say that we have two strings  $\epsilon_1$  and  $\epsilon_2$  with (\*)  $x\epsilon_1 = x$  and (\*\*)  $x\epsilon_2 = x$ . (Assumption)
2. We can choose anything as an example of  $x$ , so let's let  $x$  be  $\epsilon_2$ .
3. Putting  $\epsilon_2$  for  $x$  in (\*) we get  $\epsilon_2\epsilon_1 = \epsilon_2$
4. Putting  $\epsilon_2$  for  $x$  in (\*\*) we get  $\epsilon_2\epsilon_2 = \epsilon_2$
5. Combining the last two equations, we have:  $\epsilon_2\epsilon_1 = \epsilon_2 = \epsilon_2\epsilon_2$
6. We can apply the compositionality principle to  $\epsilon_2\epsilon_1 = \epsilon_2\epsilon_2$ , getting  $\epsilon_1 = \epsilon_2$ , which is what we want to prove.

### Chapter 2, Exercise 35, Parts a) and c)

a)  $\{a, b\}$

c)  $\{a, aa, aaa, ab, aab, aaab, abb, aabb, aaabb, abbb, aabbb, aaabbb\}$

Chapter 2, Exercise 36, Part c)

(b)  $a \quad b \quad aab$   
 $bab \quad aababab$

(d)  $\epsilon \quad ab \quad abab$   
 $ababab \quad abababab$

Chapter 2 Exercise 37

Let  $X, Y$  and  $Z$  be sets of strings over  $\{a, b\}$ .  
Prove that  $\{a\} \circ (Y \cap Z) = (\{a\} \circ Y) \cap (\{a\} \circ Z)$ .

First, let  $x \in \{a\} \circ (Y \cap Z)$ .

Then (using the definition of  $\circ$ ),  $x = ax'$ , where  $x' \in Y \cap Z$ .

Then (using the definition of  $\cap$ ),  $x' \in Y$  and  $x' \in Z$ .

So (using the definition of  $\circ$ ),  $ax' \in \{a\} \circ Y$  and  $ax' \in \{a\} \circ Z$ .

Then (using the definition of  $\cap$ ),  $x \in (\{a\} \circ Y) \cap (\{a\} \circ Z)$ .

This shows that  $\{a\} \circ (Y \cap Z) \subseteq (\{a\} \circ Y) \cap (\{a\} \circ Z)$ .

Second, let  $x \in (\{a\} \circ Y) \cap (\{a\} \circ Z)$ .

Then (using the definition of  $\cap$ ),  $x \in \{a\} \circ Y$  and  $x \in \{a\} \circ Z$ .

So (using the definition of  $\circ$ ),  $x = ax_1$ , where  $x_1 \in Y$  and  $x = ax_2$ , where  $x_2 \in Z$ .

Then, by decomposibility,  $x_1 = x_2$ .

So (letting  $x' = x_1 = x_2$ ),  $x = ax'$ , where  $x' \in Y$  and  $x' \in Z$ .

Then (using the definition of  $\cap$ ),  $x \in \{a\} \circ Y$  and  $x \in \{a\} \circ Z$ .

So (using the definition of  $\circ$ ),  $x \in \{a\} \circ (Y \cap Z)$ .

This shows that  $(\{a\} \circ Y) \cap (\{a\} \circ Z) \subseteq \{a\} \circ (Y \cap Z)$ .

Therefore  $\{a\} \circ (Y \cap Z) = (\{a\} \circ Y) \cap (\{a\} \circ Z)$ .

## Chapter 2 Exercise 40

In question 39 you only need to provide one example of a string that is in one of the sets but not the other. This is how you could answer the similar problem 40:

The following generalization is false.

$$\begin{aligned} &\text{For all sets of strings } X, Y \text{ and } Z \text{ over } \{a, b\}, \\ &X \cup (Y \circ Z) = (X \cup Y) \circ (X \cup Z). \end{aligned}$$

Find a counterexample to it.

Answer: (There are many possibilities. Here is one.) Let  $X = \{a\}$ ,  $Y = \{b\}$  and  $Z = \{b\}$  Then  $aa \in (X \cup Y) \circ (X \cup Z)$  but  $aa \notin X \cup (Y \circ Z)$

## Chapter 2 Exercise 44

Here is how you write the graph in example 57 as a data structure:

$\langle \{ \langle \text{Bigfork}, \text{Condon} \rangle, \langle \text{Condon}, \text{Bigfork} \rangle, \langle \text{Ronan}, \text{Bigfork} \rangle, \langle \text{Bigfork}, \text{Ronan} \rangle, \langle \text{Condon}, \text{Clearwater Junction} \rangle, \langle \text{Clearwater Junction}, \text{Condon} \rangle, \langle \text{Ronan}, \text{Missoula} \rangle, \langle \text{Missoula}, \text{Ronan} \rangle, \langle \text{Missoula}, \text{Clearwater Junction} \rangle, \langle \text{Clearwater Junction}, \text{Missoula} \rangle, \langle \text{Lincoln}, \text{Clearwater Junction} \rangle, \langle \text{Clearwater Junction}, \text{Lincoln} \rangle \} \rangle$

## Chapter 2, Exercise 48, Parts a), b), and d)

(a)  $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3, f(4) = 4, f(5) = 5, f(6) = 6.$

(b)  $f(0) = 1, f(1) = 2, f(2) = 4, f(3) = 8, f(4) = 16, f(5) = 32, f(6) = 64.$

(d)  $f(0) = 1, f(1) = 0, f(2) = 1, f(3) = 0, f(4) = 1, f(5) = 0, f(6) = 1.$

### Exercise:

Give a definition of multiplication by  $a$  in terms of addition of  $a$ . That is, give a recursive definition of  $f(x) = a \cdot x$  from  $g(z) = x+a$  (plus the constant function  $k(z) = 0$ ).

Answer: Base clause:  $a \cdot 0 = 0$ ; Recursion clause:  $a \cdot (n + 1) = a \cdot n + a$

## Chapter 2, Exercise 41

Show that for all sets of strings  $X$ ,  $X^{**} = X^*$ .

Proof: As usual, we break the problem into two parts, proving containment both ways.  $\Leftarrow$ :

$X^* \subseteq X^{**}$  follows trivially from the definition of  $*$ , since in general, for any  $A$ ,  $A \subseteq A^*$ .

$\Rightarrow$

This direction is a proof by induction (on the length of the string (= number of symbols in the string.)

**Note:** you don't have to use induction here, and in this case it is a bit inelegant this way, because the argument of the inductive step follows directly from properties of the Kleene star. I'm writing it this way as an example of an argument in inductive form.

*We want to show:  $X^{**} = X^*$*

Pick a string  $s \in X^{**}$

**Base case:** Say that the string has no characters. That is, it is the empty string  $\epsilon$

Then it is in both  $X^{**}$  and  $X^*$  since  $\epsilon$  is in the Kleene star of every set.

**Induction Step** Assume that  $s$  is a string of length  $n > 0$ , and for every string  $s'$  of length less than  $n$ , the following holds: If  $s' \in X^{**}$  then  $s' \in X^*$ .

Since  $s \in X^{**}$ ,  $s = s_1 s_2 \dots s_m$  where  $s_1 \in X^*$ ,  $s_2 \in X^*$ , ...  $s_m \in X^*$

Each of the strings  $s_1, s_2 \dots s_m$  is *shorter than*  $s$  and so the induction hypothesis applies.

Thus, each  $s_1 \in X^*$ ,  $s_2 \in X^*$ , ...  $s_m \in X^*$

Since each of these are in  $X^*$ , they decompose further. For every  $i$  there are strings  $s_j^i \in X$ , such that  $s_i = s_1^i s_2^i \dots s_m^i$

Thus  $s$  is a string arising from the concatenation of these  $s_j^i$ 's, so it is a concatenation of strings in  $X$ .

That is,  $s \in X^*$

Chapter 2, Exercise 48  $\{\langle Paul, Susan \rangle, \langle Lucy, Susan \rangle, \langle Lucy, John \rangle,$   
 $\langle Max, John \rangle, \langle Faye, Fred \rangle, \langle Susan, Ann \rangle,$   
 $\langle John, George \rangle, \langle George, Martha \rangle, \langle Martha, Dave \rangle,$   
 $\langle Fred, Bob \rangle, \langle Ann, Bob \rangle, \langle Ann, Carol \rangle, \langle Mike, Carol \rangle\}$