Philosophy 303
Symbolic Logic
Winter, 2009

Solved problems for Problem set 10

1. Say we add to our system the rule of Modus Tollens:

j	$P \to Q$	
	$\vdots$	
k	$\neg Q$	
	$\vdots$	
l	$\neg P$	j, k *modus tollens*

   Proof: We just need to add to the proof of Metatheorem 1 an additional case in the Base Case and an additional case in the induction step to cover the new rule. Both are essentially the same; the addition for the induction case would run like this:

   Say that $\neg P$ is written down at step n and justified by *modus tollens*. Then the premises are $P \to Q$ and $\neg Q$ for some Q. By the induction hypothesis, the $\Gamma \vdash P \to Q$ and $\Gamma \vdash \neg Q$ By the truth tables for $\to$ and $\neg$ (an answer on a problem set should draw the tables) any assignment that gives T to $P \to Q$ and $\neg Q$ must also give T to $\neg P$.

2. Chapter 7 Question 8

   Show that a set Γ of formulas of $\mathcal{L}^{\mathrm{B}}(\mathbf{V})$ is $\mathbf{V}$-saturated if and only if 1) Γ is deductively closed 2) for some formula A of $\mathcal{L}^{\mathrm{B}}(\mathbf{V})$, $A \notin \Gamma$ 3) for all formulas A and B of $\mathcal{L}^{\mathrm{B}}(\mathbf{V})$, if $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$.

   Proof: $\Rightarrow$ Assume Γ is $\mathbf{V}$-saturated.

   1) Say that $\Gamma \vdash A$. Since Γ is $\mathbf{V}$-saturated, either $A \in \Gamma$ or $\neg A \in \Gamma$. Γ is consistent, and $\Gamma \vdash A$, so $\Gamma \nvdash \neg A$, which means that $\neg A \notin \Gamma$, so $A \in \Gamma$.

   2) for some formula A of $\mathcal{L}^{\mathrm{B}}(\mathbf{V})$, $A \notin \Gamma$ follows immediately from the consistency of Γ

   3)Say that $A \vee B \in \Gamma$. If $A \in \Gamma$ or $B \in \Gamma$ we are done, so assume (for a contradiction) $A \notin \Gamma$ and $B \notin \Gamma$. Then since Γ is $\mathbf{V}$-saturated, $\neg A \in \Gamma$ and $\neg B \in \Gamma$. It is straightforward to derive a contradiction from $\{\neg A, \neg B, A \vee B\}$ (left as an exercise), but this contradicts the $\mathbf{V}$-saturatedness of Γ. Thus it cannot be that $A \notin \Gamma$ and $B \notin \Gamma$, so we're done.

$\Leftarrow$ Say that 1) Γ is deductively closed 2) for some formula A of $\mathcal{L}^{\mathrm{B}}(\mathbf{V})$, $A \notin \Gamma$ 3) for all formulas A and B of $\mathcal{L}^{\mathrm{B}}(\mathbf{V})$, if $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$.

To show that Γ is $\mathbf{V}$-saturated we have to show two things:
a) Γ is consistent, since a deductively closed set that is inconsistent contains every formula, so Γ couldn't satisfy 2) if it were inconsistent.

b) For every $A \in \mathcal{L}^{\mathrm{B}}(\mathbf{V})$, $A \in \Gamma$ or $\neg A \in \Gamma$.
Note that since $P \vee \neg P$ has a categorical derivation, $\emptyset \vdash P \vee \neg P$, and so, since $\emptyset \subseteq \Gamma$, $\Gamma \vdash P \vee \neg P$. By deductive closure, $P \vee \neg P \in \Gamma$, so by 3), $P \in \Gamma$ or $\neg P \in \Gamma$.

3. Chapter 7 Question 10

Let Γ be a set of formulas of **V** which is deductively closed (if $\Gamma \vdash A$ and $A \in \mathbf{V}$, then $A \in \Gamma$). Show that if for all parameters P of **V** either $P \in \Gamma$ or $\neg P \in \Gamma$ (and not both), then Γ is **V**-saturated.

The basic idea of the proof is to use the condition for parameters P as the base case of an induction. We'll use induction on the complexity of A to show that for all formulas A of **V**, $A \in \Gamma$ or $\neg A \in \Gamma$.

Base case:
Say that A is a sentence with no connectives. Then A must be a single parameter, and so by the condition of the theorem, $A \in \Gamma$ or $\neg A \in \Gamma$ and not both.

Induction step: Say that for every formula A' with k or fewer connectives, $A' \in \Gamma$ or $\neg A' \in \Gamma$ and not both. Say that A has k+1 connectives. Then we have these possibilities:

i) $A = \neg A_1$
ii) $A = A_1 \rightarrow A_2$
iii) $A = A_1 \vee A_2$
iv) $A = A_1 \wedge A_2$

First this observation: Choose some parameter P. Not both $P \in \Gamma$ and $\neg P \in \Gamma$, by the conditions of the theorem. But this means that for any formula A, not both $A \in \Gamma$ and $\neg A \in \Gamma$, since if Γ contained both A and $\neg A$, P and $\neg P$ would both be derivable from Γ, and since Γ is deductively closed this would mean P and $\neg P$ would both be *in* Γ, which we know not to be true.

i) Say that $A = \neg A_1$. Then by the induction hypothesis, $A_1 \in \Gamma$ or $\neg A_1 \in \Gamma$ and not both, since A_1 has only k connectives. From A_1 we can easily derive (exercise) $\neg\neg A_1$ (i.e. $\neg A$), so by deductive closure, if $A_1 \in \Gamma$ then $\neg A \in \Gamma$. So either $A \in \Gamma$ and $\neg A \in \Gamma$ (and not both, as noted above), which is what we want to show.

ii) $A = A_1 \rightarrow A_2$. By the induction hypothesis, we have that $A_1 \in \Gamma$ or $\neg A_1 \in \Gamma$ and not both, and $A_2 \in \Gamma$ or $\neg A_2 \in \Gamma$ and not both. If $\neg A_1 \in \Gamma$ it is easy to derive $A_1 \rightarrow A_2$ and if $A_2 \in \Gamma$ it is easy to derive $A_1 \rightarrow A_2$. [Exercise] In either case, by deductive closure, $A_1 \rightarrow A_2 \in \Gamma$.
So the only case we have to worry about is if $A_1 \in \Gamma$ and $\neg A_2 \in \Gamma$. In this case it is easy to derive $\neg(A_1 \rightarrow A_2)$ from $\{A_1 \neg A_2\}$, so again by deductive closure of Γ we have $\neg(A_1 \rightarrow A_2) \in \Gamma$.
Hence in every case, $(A_1 \rightarrow A_2) \in \Gamma$ or $\neg(A_1 \rightarrow A_2) \in \Gamma$. As we noted above, we

cannot have both in Γ.

iii) $A = A_1 \vee A_2$. By the induction hypothesis, we have that $A_1 \in \Gamma$ or $\neg A_1 \in \Gamma$ and not both, and $A_2 \in \Gamma$ or $\neg A_2 \in \Gamma$ and not both. If $A_1 \in \Gamma$ it is easy to derive $A_1 \vee A_2$ and if $A_2 \in \Gamma$ it is easy to derive $A_1 \vee A_2$. [Exercise] In either case, by deductive closure, $A_1 \vee A_2 \in \Gamma$.
So the only case we have to worry about is if $\neg A_1 \in \Gamma$ and $\neg A_2 \in \Gamma$. In this case it is easy to derive $\neg(A_1 \vee A_2)$ from $\{\neg A_1 \neg A_2\}$, so again by deductive closure of Γ we have $\neg(A_1 \vee A_2) \in \Gamma$.
Hence in every case, $(A_1 \vee A_2) \in \Gamma$ or $\neg(A_1 \vee A_2) \in \Gamma$. As we noted above, we cannot have both in Γ.

iv) $A = A_1 \wedge A_2$. By the induction hypothesis, we have that $A_1 \in \Gamma$ or $\neg A_1 \in \Gamma$ and not both, and $A_2 \in \Gamma$ or $\neg A_2 \in \Gamma$ and not both. If $\neg A_1 \in \Gamma$ it is easy to derive $\neg(A_1 \wedge A_2)$ and if $\neg A_2 \in \Gamma$ it is easy to derive $\neg(A_1 \wedge A_2)$. [Exercise] In either case, by deductive closure, $\neg(A_1 \wedge A_2) \in \Gamma$.
So the only case we have to worry about is if $A_1 \in \Gamma$ and $A_2 \in \Gamma$. In this case it is easy to derive $A_1 \wedge A_2$ from $\{A_1 A_2\}$, so again by deductive closure of Γ we have $A_1 \wedge A_2 \in \Gamma$.
Hence in every case, $(A_1 \wedge A_2) \in \Gamma$ or $\neg(A_1 \wedge A_2) \in \Gamma$. As we noted above, we cannot have both in Γ.

This completes the induction step, so by induction, we have that for every formula $A \in \mathbf{V}$, $A \in \Gamma$ or $\neg A \in \Gamma$. Since Γ was shown above to be consistent, we have that Γ is $\mathbf{V}$-saturated.

4. Chapter 7 Question 12

   i) A is $(B \wedge C)$, we know by the satisfaction rule for conjunction that $I(A) = \mathrm{T}$ if and only if $I(B) = \mathrm{T}$ and $I(C) = \mathrm{T}$. But by the hypothesis of induction, $I(B) = \mathrm{T}$ if and only if $B \in \Gamma$ and $I(C) = \mathrm{T}$ if and only if $C \in \Gamma$. But by Metatheorem 30 this holds if and only if $(B \wedge C) \in \Gamma$; i.e., if and only if $A \in \Gamma$. Therefore, $I(A) = \mathrm{T}$ if and only if $A \in \Gamma$.

   ii) A is $(B \vee C)$, we know by the satisfaction rule for disjunction that $I(A) = \mathrm{T}$ if and only if $I(B) = \mathrm{T}$ or $I(C) = \mathrm{T}$. But by the hypothesis of induction, $I(B) = \mathrm{T}$ if and only if $B \in \Gamma$ and $I(C) = \mathrm{T}$ if and only if $C \in \Gamma$. But by Metatheorem 31 this holds if and only if $(B \vee C) \in \Gamma$; i.e., if and only if $A \in \Gamma$. Therefore, $I(A) = \mathrm{T}$ if and only if $A \in \Gamma$.