Classical and Semirelativistic Magnetohydrodynamics with Anisotropic Ion Pressure

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Abstract

We study the magnetohydrodynamics (MHD) equations with anisotropic ion pressure and isotropic electron pressure under both the classical and semirelativistic approximations in order to develop a numerical model. The dispersion relation as well as the characteristic wave speeds are derived. In addition to the exact wave speed solutions, we also provide efficient approximate formulas for the semirelativistic magnetosonic speeds. The equations are discretized with the Rusanov and Harten-Lax-van Leer numerical schemes and implemented into the BATS-R-US MHD code. We perform a set of verification tests.

1. Introduction

To describe real magnetized plasma, magnetohydrodynamics (MHD) has been widely used in many applications by both the modeling and theoretical communities. There are various types of MHD approximations based on different assumptions and simplifications, which can capture the physical processes of interest. Such processes include, but are not limited to, collisions between particles, Hall current, electric current dissipation, heat conduction, as well as the pressure anisotropy. Pressure anisotropy arises naturally in a low density magnetized plasma, where the gyration and the field-aligned motion of the particles are not coupled by collisions. The magnetic field provides the preferred orientation, while particle collisions tend to drive the plasma isotropic by evenly distributing the parallel and perpendicular momenta with respect to the magnetic field. Without enough collisions, the parallel and perpendicular pressures can be different, however the difference is bounded by instabilities including the firehose, mirror and proton-cyclotron
instabilities [1, 2, 3]. Space plasmas, our primary interest, are basically collisionless, which means that the pressure anisotropy could play an important role.

MHD with anisotropic pressure was first investigated by Chew, Goldberger and Low [4]. They start from the Boltzmann equation and obtain the Chew-Goldberger-Low (CGL) approximation, also known as the double-adiabatic model, which is valid for single-fluid collisionless plasma with strong magnetic field and neglects the pressure transport along magnetic field lines. Later on Hau et al. [5, 6] proposed the double-polypotropic model as a more generalized description, which recovers the CGL model as a limiting case. We derive our transport equations by taking the moments of the generalized kinetic equation presented by Gombosi et al. [7]. We include the electron pressure as well, which is assumed to be isotropic. This assumption is valid in most space plasma applications, since electrons respond to perturbations much more rapidly than ions due to their small mass, as a result their momentum distribution remains approximately isotropic.

As an important extension to the classical (non-relativistic) case, we study the semirelativistic formulation. The semirelativistic approximation assumes that the plasma flow speed and the sound speed are nonrelativistic, while the Alfvén speed is relativistic. This is applicable for the case when the classical Alfvén speed is comparable or even larger than the speed of light, for example in Jupiter’s and Saturn’s magnetospheres due to strong planetary magnetic fields. For problems with moderate Alfvén speeds, the semirelativistic form of MHD equations is still useful because it can accelerate numerical convergence to steady state solutions by artificially reducing the speed of light, which is known as the ”Boris correction” in the space plasma modeling community [8]. For single-fluid ideal MHD, the semirelativistic equation set as well as characteristic waves were presented in [9].

This is the first time that a numerical model is built to solve the semirelativistic MHD equations with anisotropic ion pressure and isotropic electron pressure. As a first step, we derive the dispersion relation and solve for the characteristic wave speeds. The maximum wave propagation speed determines the maximum stable explicit time step according to the Courant-Friedrichs-Lewy (CFL) stability condition. The maximum wave speed is also required for the Rusanov (or local Lax-Friedrichs) scheme [10], while the fastest left and right wave speeds are needed for the Harten-Lax-van Leer (HLL) scheme [11]. The anisotropic MHD equations are implemented into the BATS-R-US MHD code [12, 13], which can solve various forms of
the MHD equations including Hall, semirelativistic, multi-species, multi-fluid and so on. The pressure anisotropy is the latest capability of the BATS-R-US code.

The paper first presents the MHD equations for both classical and semirelativistic cases with anisotropic ion pressure and isotropic electron pressure. In Section 3 the characteristic waves are explored for the semirelativistic approximation. The classical case and the case without electron pressure are also obtained. Section 4 describes the numerical method. In Section 5, we present verification tests using the BATS-R-US code. Section 6 contains our conclusions and plans for future work.

2. Equations

In the presence of anisotropic ion pressure and isotropic electron pressure, the pressure tensor can be written as \[4, 14]\)

\[
P = (p_\perp + p_e)\mathbf{I} + (p_\parallel - p_\perp)\mathbf{b}\mathbf{b}
\]

where \(\mathbf{I}\) is the identity tensor and \(\mathbf{b} = \mathbf{B}/|\mathbf{B}|\) is the unit vector along the magnetic field \(\mathbf{B}\). We define \(B = |\mathbf{B}|\) as the magnitude of the magnetic field for later use. The electron pressure is denoted by \(p_e\), while \(p_\parallel\) and \(p_\perp\) describe the parallel and perpendicular ion pressure components with respect to the magnetic field. The average ion scalar pressure thus can be expressed as

\[
p = \frac{2p_\perp + p_\parallel}{3}
\]

which is the trace of the ion pressure tensor divided by 3.

2.1. Non-relativistic Equations

We start with the equation set for non-relativistic MHD in the primitive-variable form

\[
\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho + \rho (\nabla \cdot \mathbf{u}) = 0
\]

\[
\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla (p_\perp + p_e) + \nabla \cdot [(p_\parallel - p_\perp)\mathbf{b}\mathbf{b}]
\]

\[
+ \frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B}) = 0
\]

\[
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times [- (\mathbf{u} \times \mathbf{B})] = 0
\]
\[ \frac{\partial p_{\parallel}}{\partial t} + (\mathbf{u} \cdot \nabla) p_{\parallel} + p_{\parallel} (\nabla \cdot \mathbf{u}) + 2p_{\parallel} \mathbf{b} \cdot (\mathbf{b} \cdot \nabla) \mathbf{u} = 0 \]  
(6)

\[ \frac{\partial p_{\perp}}{\partial t} + (\mathbf{u} \cdot \nabla) p_{\perp} + 2p_{\perp} (\nabla \cdot \mathbf{u}) - p_{\perp} \mathbf{b} \cdot (\mathbf{b} \cdot \nabla) \mathbf{u} = 0 \]  
(7)

\[ \frac{\partial p_e}{\partial t} + (\mathbf{u} \cdot \nabla) p_e + \frac{5}{3} p_e (\nabla \cdot \mathbf{u}) = 0 \]  
(8)

where \( \rho \) and \( \mathbf{u} \) represent the density and velocity, \( \mu_0 \) is the permeability of vacuum, and the polytropic index is taken to be 5/3. Note that we assume that the ion and electron velocities are equal, thus we do not consider Hall MHD for this study. Also, the collision terms which describe the interactions between ions and electrons as well as wave scatterings are all neglected. Therefore, we are dealing with an 'ideal' three-temperature MHD approximation, i.e., considering the ion parallel pressure, ion perpendicular pressure and electron pressure separately.

Compared to the isotropic MHD equations, the continuity equation (3) and the induction equation (5) remain the same. The momentum equation (4) contains the pressure tensor (1) instead of the scalar pressure in the isotropic case. The ion pressure components have their individual evolution equations (6) and (7). In the absence of collision terms, the ratio between the two pressure components might achieve unrealistic values. When implementing the equations into BATS-R-US, we add a relaxation term to the right-hand-sides of (6) and (7) to limit the ion pressure anisotropy to the range allowed by the various instabilities. This will be discussed later.

For the convenience of implementation into BATS-R-US, we adopt the average ion pressure \( \bar{p} \) as one of our primitive variables, and solve

\[ \frac{\partial \bar{p}}{\partial t} + 2\bar{p} (\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla) \bar{p} - \frac{1}{3} p_i (\nabla \cdot \mathbf{u}) + (p_i - \bar{p}) \mathbf{b} \cdot (\mathbf{b} \cdot \nabla) \mathbf{u} = 0 \]  
(9)

which is obtained by linearly combining the parallel (6) and perpendicular (7) pressure equations according to relation (2). We use \( p_i \) and \( \bar{p} \) as the primary variables, and \( p_\perp \) is obtained as \( p_\perp = (3p - p_i)/2 \).

2.2. Conservative Form

The conservative form of the equations is required in order to capture correct jump conditions across discontinuities, for instance, the Earth’s bow shock. We have density \( \rho \), momentum \( \rho \mathbf{u} \), magnetic field \( \mathbf{B} \), and total energy density

\[ e = \frac{\rho u^2}{2} + \frac{\mathbf{B}^2}{2\mu_0} + \frac{3}{2} (p + p_e) \]  
(10)
as conservative variables, and the conservative equations can be written as

\[
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \left[ \rho \mathbf{u} \mathbf{u} + p_{\perp} \mathbf{I} + p_{e} \mathbf{I} + (p_{\parallel} - p_{\perp}) \mathbf{b} \mathbf{b} - \frac{1}{\mu_0} \left( \mathbf{B} \mathbf{B} - \frac{B^2}{2} \mathbf{I} \right) \right] = 0 \quad (11)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \left[ (\mathbf{u} \times \mathbf{B}) \right] = 0 \quad (12)
\]

\[
\frac{\partial e}{\partial t} + \nabla \cdot \left[ \mathbf{u} \left( e + p_{\perp} + p_{e} + \frac{B^2}{2\mu_0} \right) + \mathbf{u} \cdot ((p_{\parallel} - p_{\perp}) \mathbf{b} \mathbf{b} - \frac{B\mathbf{B}}{2\mu_0}) \right] = 0 \quad (13)
\]

The parallel and electron pressure equations (6) and (8) are still needed to calculate \( p_{\parallel} \) and \( p_{e} \). This means that the jump conditions across a discontinuity cannot be fully determined from the conservation relations, because we do not know how to distribute the total thermal energy (obtained from the conservative quantity \( e \)) among the parallel ion, perpendicular ion and electron pressures. The problem of insufficient jump relations for discontinuities in an anisotropic plasma has been discussed by many earlier studies [15, 16, 17], in which various additional assumptions were applied to supplement the jump relations. Hudson [18] discussed types of discontinuities for an anisotropic plasma, and pointed out that the assumptions made for a general case may not be suitable for discontinuities in the solar wind. There are also detailed analysis of Rankine-Hugoniot relations modified by the pressure anisotropy for shocks in space [19, 20]. A relatively new solution to the problem was proposed by Erkaev and Vogl et al. [21, 22, 23], in which mirror and firehose instability criteria were applied to constrain the ratio between parallel and perpendicular pressures downstream of the shock. We apply a similar approach of utilizing instabilities.

2.3. Instabilities

The three types of instabilities that we consider are the firehose, mirror and proton cyclotron instabilities that can be present in plasmas with pressure anisotropy. The firehose instability [24, 25, 26] arises when the parallel pressure is large enough so that

\[
\frac{p_{\parallel}}{p_{\perp}} > 1 + \frac{B^2}{\mu_0 p_{\perp}} \quad (15)
\]
The mirror and proton cyclotron instabilities arise when the perpendicular pressure is sufficiently large. These two instabilities, especially their behaviors in space plasma regimes have been investigated by many researchers [27, 28, 29]. The criterion for the mirror instability is

$$\frac{p_\perp}{p_\parallel} > 1 + \frac{B^2}{2\mu_0 p_\perp}$$

(16)

For the proton cyclotron instability, the general form can be written as

$$\frac{p_\perp}{p_\parallel} > 1 + C_1 \left( \frac{B^2}{2\mu_0 p_\parallel} \right)^{C_2}$$

(17)

where $C_1$ and $C_2$ are constants, which vary from study to study. Anderson et al. [30] gave $C_1 = 0.85$ and $C_2 = 0.48$, while Gary et al. [31] presented three different sets of constants based on three growth rates. Also, Denton et al. [32] extracted the threshold from measurements. In our code we use $C_1 = 0.3$ and $C_2 = 0.5$ for magnetosphere simulations as it gives reasonable agreement with measurements.

When the instability criteria are satisfied, the strong wave-particle interaction tends to drive the system towards isotropy. The ratio of perpendicular to parallel pressure of a stable plasma is limited by the lower bound provided by the firehose stability threshold (15) and the upper bound provided by the mirror and proton cyclotron stability criteria (16) and (17). The incomplete equation set in conservation form is thus augmented by instability thresholds, which constrain the solution to be physically reasonable.

It is important to note that the mirror and proton cyclotron instabilities are kinetic phenomena that are not captured by the anisotropic MHD equations. The firehose instability is correctly represented by the fluid equations, but its growth rate is limited by the grid resolution. To take into account the effect of these instabilities, we add a right-hand-side source term to (6) if any of the instability conditions is fulfilled [33, 34, 35]:

$$\frac{\delta p_\parallel}{\delta t} = \frac{\overline{p}_\parallel - p_\parallel}{\tau}$$

(18)

where $\overline{p}_\parallel$ is the marginally stable parallel pressure obtained from (15), (16) or (17), respectively. This source term relaxes the anisotropy towards a stable state at a time rate $\tau$. The value of $\tau$ depends on the growth rate of instabilities, which in turn depends on the plasma state as well as the spatial
scales. Currently we explicitly set $\tau$ to a value that is small relative to the typical dynamical time scales of the problem.

The source term is applied in a split manner at the end of the time step and it is discretized point-implicitly for the sake of numerical stability as

$$p_i^{n+1} = p_i^n + \frac{\Delta t}{\tau} (\overline{p}_i - p_i^{n+1})$$  \hspace{1cm} (19)

which can be solved for $p_i^{n+1}$ to obtain the update

$$p_i^{n+1} = p_i^n + \frac{(\overline{p}_i - p_i^n)\Delta t}{\Delta t + \tau}$$  \hspace{1cm} (20)

where $\Delta t$ is the time step, and time levels $n$ and $n+1$ correspond to the incomplete and final updates, respectively.

2.4. Semirelativistic Equations

To obtain the semirelativistic form, we follow the steps in [9] for semirelativistic MHD with isotropic pressure. We rederive the equations while keeping the electric force in the momentum equation and the displacement current in Ampere’s law. Only the momentum equation (4) needs to be modified from the classical case, which can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \gamma_A^2 (\mathbf{I} + \frac{V_A^2}{c^2} \mathbf{bb}) \cdot \left[ (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{\rho} \nabla (p_{\perp} + p_e) + \frac{1}{\rho} \nabla \cdot [(p_{\parallel} - p_{\perp}) \mathbf{bb}] \right] + \gamma_A^2 \frac{1}{\mu_0 \rho} \mathbf{B} \times \left[ \nabla \times \mathbf{B} - \frac{1}{c^2} \mathbf{u} \times (\nabla \times \mathbf{E}) - \frac{1}{c_0^2} \mathbf{u} \nabla \cdot \mathbf{E} \right] = 0$$  \hspace{1cm} (21)

where the Alfvén factor

$$\gamma_A = \frac{1}{\sqrt{1 + \frac{V_A^2}{c^2}}}$$  \hspace{1cm} (22)

was introduced with the classical Alfvén speed $V_A = B/\sqrt{\mu_0 \rho}$. $c_0$ is the true value of the speed of light, while $c$ is the artificially lowered speed of light. The electric field $\mathbf{E}$ can be obtained from Ohm’s law

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B}$$  \hspace{1cm} (23)

Hence, our primitive-variable equation set of the semirelativistic MHD with anisotropic ion pressure and isotropic electron pressure is composed of
equations (3), (5) - (8) and (21). Note that when \( V_A^2 \) is much smaller than \( c^2 \), the equation set reduces to the classical form. In equation (21), the last term \( \frac{1}{c^2} \mathbf{u} \nabla \cdot \mathbf{E} \) can be dropped because that it is basically \( u^2/c^2 \) smaller than the \( \nabla \times \mathbf{B} \) term, and \( u^2 \ll c^2 \) is true for the semirelativistic limit. This simplification will be used for the derivation of characteristic waves.

In the conservative form of semirelativistic MHD, both the momentum and energy equations are different from the non-relativistic equations. The momentum equation (12) is replaced with

\[
\frac{\partial [\rho \mathbf{u} + \mathbf{E} \times \mathbf{B}/(\mu_0 c^2)]}{\partial t} + \nabla \cdot \left[ \rho \mathbf{u} \mathbf{u} + (p_\perp + p_e) \mathbf{I} + (p_i - p_\perp) \mathbf{b} \mathbf{b} + \frac{B^2 \mathbf{I}}{2\mu_0} + \frac{E^2 \mathbf{I}}{2\mu_0 c^2} - \frac{\mathbf{BB}}{\mu_0} - \frac{\mathbf{EE}}{\mu_0 c^2} \right] = 0
\]

and the energy equation (14) changes to

\[
\frac{\partial [\epsilon + E^2/(2\mu_0 c^2)]}{\partial t} + \nabla \cdot \left[ \mathbf{u} \left( \epsilon + p_\perp + p_e - \frac{B^2}{2\mu_0} \right) + (p_i - p_\perp) \mathbf{u} \cdot \mathbf{b} \mathbf{b} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right] = 0
\]

where \( E = |\mathbf{E}| \) is the magnitude of the electric field.

3. Characteristic Wave Speeds

To find the characteristic waves, we need to solve for the eigenvalue problem of a set of nine equations. We solve the semirelativisitic case, and then obtain the classical limit by taking \( \gamma_A = 1 \). In order to obtain the characteristic matrix, we write the equation set in a quasi-linear form in one dimension:

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{M}_x \frac{\partial \mathbf{U}}{\partial x} = 0
\]

The variables \( \mathbf{U} = (\rho, \mathbf{u}, \mathbf{B}, p_i, p_\perp, p_e) \) only depend on \( x \) and \( t \). We utilize the MAPLE software to extract the coefficients from equations (3), (5) - (8) and (21), and form the 9×9 characteristic matrix. Because the characteristic wave speeds do not depend on the coordinate system, we simplify our problem by setting \( B_z = 0 \), i.e., the coordinate system is rotated such that the magnetic
field lies in the $x-y$ plane. The final characteristic matrix $\mathbf{M}_x$ can be written as

$$
\begin{pmatrix}
u & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \gamma_A^2 u_x + \chi_{11} & \chi_{12} & 0 & \kappa_1 & 0 & \eta_{11} & \eta_{12} & \eta_{13} \\
0 & \chi_{21} & \gamma_A^2 u_x + \chi_{22} & 0 & \kappa_2 & 0 & \eta_{21} & \eta_{22} & \eta_{23} \\
0 & \chi_{31} & \chi_{32} & \gamma_A^2 u_x + \chi_{33} & \kappa_3 & \nu & 0 & 0 & 0 \\
0 & B_y & -B_x & 0 & u_x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -B_x & 0 & u_x & 0 & 0 & 0 \\
0 & p_1 (1 + 2b_x^2) & 2p_1 b_x b_y & 0 & 0 & 0 & u_x & 0 & 0 \\
0 & p_1 (2 - b_x^2) & -p_1 b_x b_y & 0 & 0 & 0 & 0 & u_x & 0 \\
0 & \frac{5}{3}p_1 & 0 & 0 & 0 & 0 & 0 & 0 & u_x \\
\end{pmatrix}
$$

(27)

where

$$
\chi = \frac{\gamma_A^2}{\mu_0 pc^2} \begin{pmatrix}
B_x^2 u_x - B_y^2 u_x & 2B_y B_x u_y & 0 \\
2B_y B_x u_y & B_y^2 u_x - B_x^2 u_x & 0 \\
-B_y^2 u_x & -B_x^2 u_y & 0 \\
\end{pmatrix}
$$

(28)

$$
\kappa = \frac{\gamma_A^2}{\mu_0 pc^2} \begin{pmatrix}
(c^2 - u_x^2)B_y + (2\mu_0 pc^2 B^{-2} + 1)B_y b_x^2 (p_1 - p_1) \rho^{-1} \\
(u_x^2 - c^2)B_x + (b_x^2 - (2u_x^2 - b_x^2)\mu_0 pc^2 B^{-2}) B_y^2 (p_1 - p_1) \rho^{-1} \\
-B_y u_x u_y \\
\end{pmatrix}
$$

(29)

$$
\nu = \frac{\gamma_A^2}{\mu_0 pc^2} \left[(u_x^2 - c^2)B_x + B_y u_x u_y - \mu_0 c^2 B_x B^{-2} (p_1 - p_1)\right]
$$

(30)

$$
\eta = \begin{pmatrix}
\gamma_A^2 b_x^2 \rho^{-1} \\
b_x b_y \rho^{-1} \\
\gamma_A^2 b_x b_y \rho^{-1} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

(31)

We use the notation $b_y = B_y/B$. With the help of MAPLE and after some complicated algebra, the characteristic equation det($\mathbf{M}_x - \mathbf{I}\lambda$) = 0 is reduced to the following form:

$$
(\lambda - u_x)^3 P_2(\lambda) P_4(\lambda) = 0
$$

(32)

where the wave speed $\lambda$ is one of the eigenvalues of $\mathbf{M}_x$, $u_x$ is the flow velocity along the direction $x$. $P_2$ and $P_4$ are second- and fourth-order polynomials:

$$
P_2 = \lambda(\lambda - u_x) + \gamma_A^2 \left[\lambda(\mathbf{u} \cdot \mathbf{b})b_x V_A^2 - u_x(\lambda - u_x) - \left(V_A^2 + \frac{p_1 - p_1}{\rho}\right)b_x^2\right]
$$

(33)
\[ P_4 = (\lambda - u_x)^4 - \left( a^2 + \frac{2p_\perp - 3p_t}{\rho} + \frac{2p_t - p_\perp b_x^2}{\rho} \right) (\lambda - u_x)^2 \]
\[ = -(c^2 - \lambda^2) \frac{V_A^2}{c^2} [(\lambda - u_x)^2 - a^2b_x^2] \]
\[ = \left[ \frac{p_\perp^2 - 3p_t p_\perp}{\rho^2} (1 - b_x^2) + \frac{3p_t^2}{\rho^2} b_x^2 - \frac{5p_e}{3\rho} \left( \frac{4p_\perp}{\rho} + p_\perp b_x^2 - 3p_t \right) \right] b_x^2 \]

where
\[ a^2 = \frac{3p_t + 5p_e}{\rho} \]

is defined as the sound speed and \( b_x = B_x / B \).

Equation (32) is the dispersion relation, whose roots \( \lambda \) describe the corresponding characteristic waves. Three of the roots are identical:
\[ \lambda_{1,2,3} = u_x \]

They represent three entropy waves related to the three pressures \( p_\parallel, p_\perp \) and \( p_e \). In the classical limit, the entropy waves remain the same.

3.1. Alfvén Waves

The roots of \( P_2 = 0 \) correspond to a pair of Alfvén waves:
\[ \lambda_{4,5} = \frac{1}{2} \gamma_A \left[ u_x - \frac{V_A^2}{c^2} (u \cdot b) b_x \right] + \frac{u_x}{2} \]
\[ \pm \sqrt{\gamma_A \left( V_{A,xx} + \frac{p_\perp - p_t b_x^2 - u_x^2}{\rho} \right) + \left[ \frac{1}{2} \gamma_A \left( u_x - \frac{V_A^2}{c^2} (u \cdot b) b_x \right) + \frac{u_x}{2} \right]^2} \]

where \( V_{A,xx} = V_A b_x = \sqrt{B_x^2 / \mu_0 \rho} \). Although the formula is quite complicated, there is no \( p_e \) dependence, which means that the presence of the electron pressure does not affect the Alfvén wave solutions. This is expected, since the electron pressure is isotropic, which can be viewed as an analog to the isotropic pressure in single-fluid MHD, and only contributes to the longitudinal (compressional) waves.

In the classical limit with \( V_A \ll c \) and \( \gamma_A \) approaching 1, the Alfvén wave speeds reduce to
\[ \lim_{V_A \ll c} \lambda_{4,5} = u_x \pm \sqrt{\frac{1}{\mu_0} \frac{B_x^2}{b_x^2} + \left( p_\perp - p_t \right) b_x^2} \]
This is consistent with Baranov’s results [36]. Comparing to isotropic MHD, pressure anisotropy enters the expression as the difference between the parallel and perpendicular pressures. This expression becomes imaginary if the condition (15) is fulfilled, and then the firehose instability will arise.

3.2. Fast and Slow Magnetosonic Waves

The remaining four roots of \( P_4 = 0 \) in (34) are associated with magnetosonic waves. The equation itself is very complicated, so are the roots. It is much simpler to look into the classical limit first, as the terms proportional to \( V_A^2/c^2 \) will vanish. The resulting equation has only fourth- and second-order terms of \( \lambda - u_x \), which can be easily solved:

\[
\lim_{V_A \ll c} \lambda_{6,7,8,9} = u_x \pm \frac{1}{\sqrt{2 p}} \left\{ \frac{B^2}{\mu_0} + 2 p_\perp + \frac{5}{3} p_e + (2 p_\parallel - p_\perp) b_x^2 \right\}^{1/2}
\]

where we substituted \( a^2 \) back from (35). The second \( \pm \) corresponds to fast(+) and slow(−) magnetosonic waves, respectively. For each type, there are two waves propagating symmetrically along opposite directions with respect to the bulk flow speed \( u_x \), corresponding to the first \( \pm \) sign. With the pressure anisotropy, the magnetosonic wave speeds are much more complicated than in the isotropic case. In the case of neglecting the electron pressure, we recover the formula obtained by Baranov [36].

Next we consider the semirelativistic case. The general case requires the solution of a general fourth-order polynomial. While this can be done, it is very expensive. To obtain an approximation, we study some special cases that have simpler solutions. First we take the zero flow velocity assumption, i.e. \( u = 0 \). We define

\[
\bar{a}^2 = a^2 \left( 1 + \frac{V_A^2}{c^2} \right) + \frac{2 p_\perp - 3 p_\parallel}{\rho} + \frac{2 p_\parallel - p_\perp}{\rho} b_x^2
\]
related to the sound speed, and
\[
b^2 = \frac{b_x^2}{\rho^2} \left[ 3p_x p_x (2 - b_x^2) - p_x^2 (1 - b_x^2) - 3p_x^2 b_x^2 - \frac{5p_e}{3} (4p_x b_x^2 - p_x b_x^2 - 3p_x) \right]
\]
(41)
to include the extra terms related to the pressure anisotropy and the direction of the magnetic field. The eigenvalues \( \lambda_{6,7,8,9} \) for no flow approximation are
\[
\lim_{u_x \to 0} \lambda_{6,7} = \pm \frac{1}{\sqrt{2}} \sqrt{\alpha^2 - V_A^2 - \sqrt{\gamma_A^4 (\alpha^2 + V_A^2)^2 - 4 \gamma_A^2 (a^2 V_A^2 + b^2)}}
\]
(42)
and
\[
\lim_{u_x \to 0} \lambda_{8,9} = \pm \frac{1}{\sqrt{2}} \sqrt{\alpha^2 + V_A^2 + \sqrt{\gamma_A^4 (\alpha^2 + V_A^2)^2 - 4 \gamma_A^2 (a^2 V_A^2 + b^2)}}
\]
(43)
corresponding to the slow and fast magnetosonic wave speeds, respectively.

Another special case is when the magnetic field is parallel to the x direction, i.e. \( b_x = 1 \). This simplifies the eigenvalues to be
\[
\lim_{b_x \to 1} \lambda_{6,7} = u_x \pm a
\]
(44)
\[
\lim_{b_x \to 1} \lambda_{8,9} = \gamma_A u_x \pm \gamma_A V_A \sqrt{1 + \frac{p_x - p_0}{V_A^2 \rho} - \gamma_A^2 \frac{u_x^2}{c^2}}
\]
(45)
The last special case we consider is when the magnetic field is perpendicular to the x direction, i.e., \( b_x = 0 \). In this case the wave speeds become
\[
\lim_{b_x \to 0} \lambda_{6,7} = u_x
\]
(46)
\[
\lim_{b_x \to 0} \lambda_{8,9} = \gamma_A^2 u_x \pm \gamma_A V_A \sqrt{\frac{2p_x + (5/3)p_e}{\rho}} + V_A^2 \left( 1 - \gamma_A^2 \frac{\beta_A^2}{c^2} \right)
\]
(47)
Based on the special cases and the approximate formulas for the isotropic case [9], we construct the following approximate slow and fast magnetosonic wave speeds for the general case:
\[
\tilde{\lambda}_{6,7} = u_x \pm \tilde{c}_x
\]
(48)
\[
= u_x \pm \frac{1}{\sqrt{2}} \sqrt{\gamma_A^2 (\alpha^2 + V_A^2) - \sqrt{\gamma_A^4 (\alpha^2 + V_A^2)^2 - 4 \gamma_A^2 (a^2 V_A^2 + b^2)}}
\]
12
and

\[ \tilde{\lambda}_{8,9} = \frac{\gamma_A^2 u_x \pm \tilde{e}_f}{\gamma_A^2 u_x \pm \frac{1}{\gamma_A^2} \sqrt{\gamma_A^4 (\gamma_A^2 + \nabla_A^2) + \sqrt{\gamma_A^4 (\gamma_A^2 + \nabla_A^2)^2 - 4 \gamma_A^2 (a^2 \nabla_{A_{,x}}^2 + b^2)}}} \]  \hspace{1cm} (49)

where we defined

\[ \nabla_A^2 = V_A^2 \left( 1 - \frac{u_x^2}{c^2} \right) \quad \text{and} \quad \nabla_{A_{,x}}^2 = V_{A_{,x}}^2 \left( 1 - \frac{u_{x,x}^2}{c^2} \right) \]  \hspace{1cm} (50)

Equations (48) and (49) reduce to the special cases in the corresponding limits. First of all, they reduce to the classical magnetosonic wave speeds (39) for the case of Alfvén speed negligible compared to the artificially reduced speed of light, i.e., \( V_A \ll c \) and \( \gamma_A \rightarrow 1, V_A \rightarrow V_A, \nabla_{A_{,x}} \rightarrow V_{A_{,x}} \). Secondly, for \( u_x = 0 \), equations (42) and (43) are recovered. Lastly, when substituting \( b_x = 1 \) or \( b_x = 0 \) for the parallel and perpendicular magnetic field limits, (48) and (49) are consistent with (44), (45), and (46), (47), respectively.

In order to check the accuracy, we compare the exact solutions of \( P_4 = 0 \) with the values given by the approximate formulas. The relative error is evaluated as the difference between the approximate and exact solutions then divided by the exact solution:

\[ \xi_{s+} = \frac{\tilde{\lambda}_6 - \lambda_6}{\lambda_6}, \quad \xi_{s-} = \frac{\tilde{\lambda}_7 - \lambda_7}{\lambda_7}, \quad \xi_{f+} = \frac{\tilde{\lambda}_8 - \lambda_8}{\lambda_8}, \quad \xi_{f-} = \frac{\tilde{\lambda}_9 - \lambda_9}{\lambda_9} \]  \hspace{1cm} (51)

We investigate the errors statistically by generating \( 10^5 \) random points in a 6-dimensional variable space of \( u_x, B, b_x, p_y, p_z, \) and \( p_v \), while fix \( \rho = 1, \ c = 1, \) and \( \mu_0 = 1 \). Recall the definition of semirelativistic MHD, the bulk flow velocity \( u_x \) and the sound speed \( a \) should always be much smaller than the speed of light \( c \), yet the Alfvén speed \( V_A \) can exceed \( c \). We thus limit variable ranges to be \( 0 < u_x < c/3, \ 0 < b_x < c/3, \ 0 < V_A < 3c, \ 0 < b_x < 1 \) and \( 0 < p_v < 5p_1 \) that covers the physically reasonable range of anisotropy. The relative errors are shown in Figure 1. For the right going fast speed \( \tilde{\lambda}_8 \), the errors are within 5%. This is the most important eigenvalue, since \( \lambda_8 \) is the largest wave speed (for \( u_x > 0 \)) that determines the explicit time step and it is also used in the Rusanov and HLL schemes. The HLL scheme also uses \( \lambda_9 \), but it does not contribute much when its value approaches zero. So we are interested in the errors of \( \lambda_9 \) away from zero, which are acceptable. The
plot of $\xi_{f+}$ also shows that the points are mainly in the positive part, which means that $\hat{\lambda}_8$ overestimates the exact fast speed in almost every case with very few exceptions. This in turn ensures that the CFL condition based on $\hat{\lambda}_8$ is sufficiently safe to maintain stability. When both $u_e$ and $a$ lie between $c/3$ and $c$, the errors are found to be within 20%. Overall, the approximate formulas are accurate enough for our purposes within the parameter range of interest.

4. Implementation

We implemented both the classical and semirelativistic MHD equations with anisotropic ion pressure and isotropic electron pressure into the BATS-R-US code. The implementation of the various temporal and spatial discretization schemes have been reported in detail during the development of BATS-R-US [12, 37, 38, 13], and here we only describe those parts of the algorithm that are specific to the anisotropic pressure.
We solve the conservative form of the continuity, momentum, induction and energy equations: (11)-(14) in the classical case, and (11), (13), (24), (25) in the semirelativistic case. The pressure equations (6), (9) and (8) cannot be cast into full conservation form, so they are solved in the following non-conservative form:

\[
\frac{\partial p_t}{\partial t} + \nabla \cdot (p_t \mathbf{u}) = -2p_t \mathbf{b} \cdot (\mathbf{b} \cdot \nabla)\mathbf{u}
\]

\[
\frac{\partial p}{\partial t} + \nabla \cdot (p \mathbf{u}) = (p - p_t)\mathbf{b} \cdot (\mathbf{b} \cdot \nabla)\mathbf{u} - \left(p - p_t \frac{\rho}{3}\right) \nabla \cdot \mathbf{u}
\]

\[
\frac{\partial p_e}{\partial t} + \nabla \cdot (p_e \mathbf{u}) = -\frac{2}{3}p_e \nabla \cdot \mathbf{u}
\]

The above equations have pure divergence terms on the left hand sides where we use the usual flux formulation of the finite volume schemes. The right hand sides of equations (52), (53) and (54) consist of the gradient and divergence of the velocity multiplied by some terms. These coefficient terms are simply taken at the cell centers, while the gradient and divergence of the velocity are evaluated analogously to the fluxes used in the conservative equations. In particular, we use a limited reconstruction procedure with second or third order accurate slope limiters to obtain the left and right states in the primitive variables. At resolution changes we use the algorithm of [39].

The left and right velocities are averaged as

\[
\bar{\mathbf{u}} = \frac{\mathbf{u}_L + \mathbf{u}_R}{2}
\]

for the Rusanov scheme, and

\[
\bar{\mathbf{u}} = \frac{\lambda_+ \mathbf{u}_L - \lambda_- \mathbf{u}_R}{\lambda_+ - \lambda_-}
\]

for the HLL scheme, where \(\lambda_-\) and \(\lambda_+\) are the most negative and most positive wave speeds, respectively. The wave speeds are calculated with the exact formula (39) in the classical case, while for the semirelativistic case, the approximate magnetosonic wave speed formulas (48) and (49) are applied. Finally, the divergence and gradient of the velocity are obtained by integrating \(\bar{\mathbf{u}}\) over the surface of the cell in the usual finite volume manner.

BATS-R-US can either solve for the energy density \(e\) or the ion pressure \(p\) or use a hybrid approach and solve for both. In the hybrid scheme the
energy equation can be used near shock waves to get the correct energy jump conditions, while the pressure equation provides more accurate and robust solution in regions where the thermal energy density is a small fraction of the total energy density. In grid cells where the energy equation is used, \( p^* \) calculated from (53) is overwritten by the pressure calculated from the energy density using (10), while the opposite is done in grid cells where the pressure equation is used. The parallel and electron pressures \( p_p \) and \( p_e \) are always computed from equations (52) and (54), respectively. Finally, the parallel pressure has to be corrected according to (20) if any of the instability conditions (15) to (17) are violated.

The full algorithm can be summarized as following:

1. Reconstruct the left and right states using limited slopes of the primitive variables.
2. Calculate the Rusanov or HLL fluxes and the face centered values of the velocity from the left and right states.
3. Calculate the divergence of the fluxes and the divergence and gradient of the velocity by integrating over the cell faces.
4. Convert the classical momentum and energy densities to the semirelativistic variables in the semirelativistic case.
5. Apply the fluxes and source terms to update the variables.
6. Convert the semirelativistic momentum and energy densities back to the classical variables in the semirelativistic case.
7. Either overwrite pressure based on energy or the other way around.
8. Calculate the instability limits.
9. Apply the relaxation term to the parallel pressure.

To achieve second order accuracy in time, we apply a two-stage Runge-Kutta scheme. We can also use an implicit or explicit/implicit time stepping algorithm as described in [38]. In all cases, the instability limits and the relaxation terms are only applied after the full time step is completed.

5. Numerical tests

To verify the code, a large set of numerical tests have been designed and carried out, including solving the relativistic and classical equations, solving for energy density or pressure, using Rusanov and HLL schemes, uniform and non-uniform meshes, as well as explicit and implicit time stepping schemes.
In this paper we present tests involving the propagation of characteristic MHD waves on a 1D mesh and the simulation of the firehose instability. We use a grid convergence study to establish the accuracy of the scheme. In all tests we use the HLL scheme and normalized the units of the magnetic field so that $\mu_0 = 1$. We do not use the pressure relaxation term for the tests.

5.1. Magnetosonic wave propagation

We have performed magnetosonic wave propagation tests for many different magnetic field orientations both for classical and semirelativistic cases. Small perturbations are applied on a uniform background state, and then the simulated wave speeds are compared with the analytic solution.

As a specific example, we describe a test with magnetosonic waves propagating perpendicular to the background magnetic field in the semirelativistic approximation. In this case, the analytic solution for the fast wave speed is given by equation (47), while the slow wave speed given by equation (46) is equal to the flow speed $u_e$. We compute on a 1D domain ranging from $x = -30$ to $x = +30$ with periodic boundary conditions, and set the uniform background as $\rho = 1$, $u_x = 12$, $u_y = u_z = 0$, $B_x = B_z = 0$, $B_y = 30$, $p_i = 14$, $p = 16$, $p_e = 12$, $c = 30$. The density, pressure, and magnetic field perturbations are sinusoidal waves centered at $x = 0$ and limited within $-3 < x < 3$: $\delta \rho = 0.01 \cos(2\pi x/12)$, $\delta p_i = 0.14 \cos(2\pi x/12)$, $\delta p = 0.16 \cos(2\pi x/12)$, $\delta p_e = 0.195 \cos(2\pi x/12)$, and $\delta B_y = 0.3 \cos(2\pi x/12)$. For each perturbation, the maximum magnitude is around 1% of the background value. A pair of excited fast waves thus propagate along $+x$ and $-x$ directions asymmetrically with velocities $\lambda_8 = 27$ and $\lambda_9 = -15$ respectively. To minimize discretization errors, we used 3,000 grid cells. Figure 2 shows the density, ion parallel pressure and electron pressure at the initial and final times. The numerical propagation speeds of the pair of fast waves agree well with the analytic values.

5.2. Alfvén wave propagation and Firehose Instability

We did Alfvén wave propagation tests similar to the magnetosonic wave tests by posing perturbations on the velocity and magnetic field. As mentioned in section 3.1, the electron pressure does not affect the Alfvén wave speed. The propagation speed was compared to the analytical solution, and we found good agreement in all tests.

Here we show a test of the firehose instability in the classical limit, which can be triggered from the Alfvén mode by increasing the parallel pressure.
Figure 2: A pair of fast waves are generated from the initial center perturbation (solid lines). The two waves move in opposite directions at speeds $-15$ and $+27$, respectively. The dotted lines show the waves at the final time $t = 1$. The vertical dashed lines show where the center of the wave should be at $t = 1$ based on the analytic formula (47).

Within the domain $-6 < x < 6$, we have a uniform background with $\rho = 1$, $u_x = u_y = u_z = 0$, $B_x = 10$, $B_y = B_z = 0$, $p_i = 104$, $p = 110/3$, $p_c = 10$. According to the wave speed formula (38), the calculated Alfvén speed is $\sqrt{-1}$. This means that the perturbations in $u_y$ and $B_y$ will be shifted by 90 degrees. We apply the following perturbations: $\delta u_y = 0.01 \cos(2\pi x/6)$ and $\delta B_y = 0.1 \cos(2\pi x/6 + \pi/2)$. Instead of propagation, the perturbations start to grow exponentially. Figure 3 shows three snapshots of the growing instability. There are small oscillations near the local extrema at $t = 0.2$. These are caused by the numerical errors that introduce short wave length perturbations which grow much faster than the original perturbation. Figure 4 shows the growth of the $y$-direction kinetic energy $E_{ky} = \rho u_y^2/2$ as a function of time. The analytical growth rate is overplot for comparison. The agreement is very good up to $t = 0.25$. After this time, the growth of the kinetic energy exceeds the theoretical rate, because the short wavelength perturbation overtake the growth trend.

5.3. Grid Convergence Tests

In order to check the accuracy of the numerical scheme, we did several grid convergence studies. The test involving circularly polarized Alfvén wave
Figure 3: Firehose instability test, growing wave pattern at $t=0$s(solid line), $t=0.1$s(dot line), and $t=0.2$s(dash line).

Figure 4: The $y$-direction kinetic energy growth against time for the firehose instability test. The growth curve is plotted as the solid line, while the dash line represents the theoretical exponential growth rate.
propagation was presented in [13].

First, we show a test for sound waves propagating along the magnetic field. For convenience we only consider the classical limit, of which the exact solution can be easily calculated. The sound wave is excited in the domain $-5 < x < 5$ of zero-flow plasma with the magnetic field parallel to the wave propagation direction: \( \rho = 1, \ u_x = u_y = u_z = 0, \ B_x = 10, \ B_y = B_z = 0, \ p_i = p = 5 \) and \( p_e = 6 \). The perturbations are of the order \( 10^{-6} \) in magnitude: \( \delta \rho = 10^{-7} \cos(2\pi x/5), \ \delta u_x = 5 \times 10^{-7} \cos(2\pi x/5), \ \delta p_i = \delta p = 1.5 \times 10^{-6} \cos(2\pi x/5), \) and \( \delta p_e = 10^{-6} \cos(2\pi x/5) \). The resulting wave is a pure sound wave with speed \( a = 5 \). The test is performed on 1D grids with 100, 200, 400, 800 and 1600 cells, and with the monotonized central (MC) and Koren slope limiters, respectively. The boundary conditions are periodic. The final time is set to \( t = 5 \), so the wave propagates 5 full wave lengths. We compare the final state to the initial perturbed state as the exact solution. The relative error is calculated for each grid resolution, which is the absolute difference between the numerical solution and the exact solution then divided by the absolute value of the exact solution. Figure 5 confirms that the scheme converges to the analytic solution at the expected convergence rate.

Second, as an interesting extension of the propagating semirelativistic magnetosonic wave test, we generate a pair of nonlinear fast waves by increasing the magnitudes of perturbations to be ten times of the ones used in the test before and using \( \cos^2 \) perturbations instead of \( \cos \), while other parameters remain the same. The nonlinear feature of the waves can be clearly seen from Figure 6. Similar to the sound wave test, this test is performed with 100, 200, 400, 800 and 1600 cells. The final state of the simulation with 3200 cells is regarded as the reference solution that other results are compared to. Figure 7 shows the convergence rates which are essentially second order. Note that the steepening of the waves creates a non-smooth solution by the end of the simulation so that second order accuracy is not necessarily expected.

6. Conclusion

We described the equation set for the non-relativistic and semirelativistic MHD with anisotropic ion pressure and isotropic electron pressure, a three-temperature plasma. The semirelativistic equation set is useful when the classical Alfvén speed is comparable or exceed the speed of light. It is also
Figure 5: Convergence study for the sound wave test. The two curves are results of the monotonized central (MC) and Koren limiters. The dashed lines indicate the second and third order convergence rates, respectively.

Figure 6: A pair of nonlinear semirelativistic fast waves are excited from the initial center perturbation (solid lines). The dotted lines show the waves at the final time $t = 1$. The plot shows the simulation with 3200 cells.
suitable for the artificially reduced speed of light often used in space plasma modeling.

To build a numerical model, we need to determine the characteristic speeds. We solve for the semirelativistic speeds first and obtain the non-relativistic solutions as a limiting case. With two more equations compared to the isotropic single-fluid MHD, the resulting dispersion relation contains nine characteristic waves. Three of them are entropy waves corresponding to three temperatures, two of them are the Alfvén waves, while the remaining four are fast and slow magnetosonic waves. We calculate the exact speed for the Alfvén wave and derive the approximate magnetosonic wave speeds for the semirelativistic case. The accuracy of the approximate formulas are demonstrated by comparing them to the exact solution. For the classical case, we obtain the exact formulas for all wave speeds.

Instabilities play an important role in limiting the anisotropy of the ion pressure. The fluid equations correctly describe the firehose instability, but the mirror and proton cyclotron instabilities can only be fully captured by a kinetic model. We use a source term in the parallel pressure equation to relax the solution towards the marginally stable state whenever any of the stability conditions are violated. The instability conditions also augment the
incomplete set of conservation laws to establish physically reasonable jump conditions across discontinuities.

In order to verify our theory and implementation into the code, we performed several numerical tests. Both the magnetosonic and the Alfvén wave speeds were checked through wave propagation tests. We also simulated the firehose instability and compared it with the analytical solution.

The anisotropic MHD code implemented into BATS-R-US has various applications. We currently apply it to the Earth’s magnetosphere and compare the simulations with measurements in order to validate the model. The detailed methodology and results will be reported elsewhere. We also plan to apply the code to model the solar corona where the anisotropic heating is essential. Future extensions of the model include the electron pressure anisotropy and the Hall term.

References


25


