A UNIFIED TRANSPORT EQUATION FOR BOTH COSMIC RAYS AND THERMAL PARTICLES

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ABSTRACT

We present a unified transport equation that is valid for particles of all energies if the particle mean free paths are much smaller than macroscopic fluid length scales. If restricted to particles with random speeds much greater than fluid flow speeds, this equation reduces to the previously discussed extended cosmic-ray transport equation. It is significant that this allows one to describe the acceleration of particles from thermal energies to cosmic-ray energies using one transport equation. This is in contrast to previous transport equations (the Parker equation and the extended cosmic-ray transport equation), which were restricted to fast particles. The close connection to the extended cosmic-ray transport equation is demonstrated.

Subject headings: acceleration of particles — cosmic rays

1. INTRODUCTION

The transport equation for energetic particles scattered by turbulent irregularities in the embedded magnetic fields of plasmas, first derived by Parker (1965), is the primary equation used in a variety of astrophysical contexts, ranging from comets to extragalactic radio sources. This equation provides a very good approximation, requiring only that the distribution function be nearly isotropic and that particle random speeds be much greater than the fluid flow speed, which is assumed to be nonrelativistic.

The restriction to fast particles is at least in part due to the desire to work in terms of a particle distribution referred to a single inertial frame. An isotropic distribution of fast particles that is connecting past an inertial observer is still approximately isotropic in the inertial frame. However, when the momentum variable in the distribution is referred to the frame of the convecting fluid, restrictions on particle speed can be relaxed. It is for this reason that we are able to present a transport equation correct for particles of arbitrary energy.

In recent years, significant modifications have been introduced to the cosmic-ray transport equation. These modifications have been studied in a series of papers including Earl, Jokipii, & Morfill (1988), Webb (1989), Jokipii, Kota, & Morfill (1989), and Jokipii & Morfill (1990). See also Berezhko & Krymsky (1981), Williams & Jokipii (1991, hereafter WJ), included the effects of an average magnetic field and gave a full derivation of the equation. Since that time, we have gained insights into the nature of the extended transport equation (ETE) that simplify not only its interpretation, but its practical application. Therefore, this Letter is presented to show how a minor change of interpretation in the ordering of small terms allows one to see how the ETE can be used for particles of all energies. This paper should be an adequate starting point for further investigations using the equation, which we believe will turn out to be quite a powerful tool for investigating particle transport and acceleration in astrophysical plasmas.

2. A TRANSPORT EQUATION FROM THE ZERO TH MOMENT OF THE KINETIC EQUATION

In WJ, the ETE was obtained by taking moments of the kinetic equation with a simple relaxation-time scattering operator. There is no point in repeating the details of that calculation here; we present merely the results. There are three quantities involving the fluid velocity $U^a(x^b, t)$ that will be used in the following discussion.

The convective time derivative:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + U^b \frac{\partial}{\partial x^b} .$$

The convective spatial derivative:

$$\frac{d}{dx^a} \equiv \frac{\partial}{\partial x^a} + U^a \frac{\partial}{\partial x^a} .$$

The traceless velocity shear tensor:

$$\Lambda_{ab} \equiv \frac{\partial U^a}{\partial x^b} + \frac{\partial U^b}{\partial x^a} - \frac{2}{3} \frac{\partial U^c}{\partial x^c} \delta_{ab} .$$

Taking moments of the equations means integrating over momentum-space solid angle and dividing by $4\pi$. For the particle distribution $f(x^a, p^a, t)$, a function of position, momentum, and time, the zeroth, first, and second moments are as follows:

$$\frac{1}{4\pi} \int f(x^a, p^a, t) d\Omega_p \equiv f_0(x^a, p, t) ,$$

$$\frac{1}{4\pi} \int p^a f(x^a, p^a, t) d\Omega_p \equiv S^a(x^a, p, t) , \quad (1)$$

$$\frac{1}{4\pi} \int p^a p^b f(x^a, p^a, t) d\Omega_p \equiv \frac{p^2}{3} f_0 \delta_{ab} + P_{ab}(x^a, p, t) \equiv T_{ab} .$$

Thus, the particle distribution is expressed as a sum of functions of the magnitude of momentum, $p$.

The kinetic equation, correct to order $U/c$ and with the relaxation scattering term, is given by

$$\frac{df}{dt} + U^a \frac{df}{dx^a} + \frac{p^a df}{m dx^a} + \left[ F^a - m \frac{dU^a}{dt} - p^b \frac{dU^a}{dx^b} \right] \frac{df}{dp^a} \equiv f_0 - f \frac{t}{\tau}. \quad (2)$$

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In this equation, the particle momenta are referred to the frame moving with velocity $U(x^j, t)$. The spatial and time coordinates are referred to an inertial frame. Using the comoving momentum will allow us to treat both slow and fast particles where, in previous treatments, transport equations were restricted to treating fast particles only.

The zeroth moment of the kinetic equation, correct to order $U/c$, is

$$
\frac{\partial f_0}{\partial t} + \frac{\partial}{\partial x^a} \left( f_0 U^a \right) + \frac{d}{dx^a} \left( \frac{S^a}{m} \right) = \frac{m}{dt} \frac{U^a}{dP^a} \frac{\partial}{\partial P^a} \left( pS^a \right) + \frac{\partial U^a}{\partial x^a} \frac{\partial}{\partial P^a} \left( pT_{ab} \right). \tag{3}
$$

A version of this equation that ignored terms of order $U/c$ was written down by Jokipii et al. (1989). See Webb (1985, 1989) for the version of this equation that allows relativistic flow speeds. Equation (3) is quite general. Since terms no higher than quadratic in $P^a$ enter the kinetic equation, then no moments of $f$ higher than the second moment enter the zeroth moment of the kinetic equation. There are no assumptions about isotropy. The only assumption with regard to scattering is that the zeroth moment of the scattering operator is zero. Approximations and assumptions will enter only when expressions for $S^a$ and $T_{ab}$ are obtained in terms of $f_0$.

3. ORDERING AND THE HIGHER MOMENTS

To obtain a transport equation in terms of the isotropic part of the distribution, $f_0$, $S^a$, and $P_{ab}$ must be obtained in terms of $f_0$. Then, if the particle distribution is nearly isotropic, this equation for $f_0$ will approximate the evolution of $f$. Ideally, one would solve exactly the first moment equation for $S^a$ and the second moment for $P_{ab}$. But these equations are coupled with each other and are quite complicated. Since we are interested only in distributions that are nearly isotropic, $S^a < P_f$ and $P_{ab} < P_f^2$. Thus we can obtain expressions for $S^a$ and $P_{ab}$ correct to some power of a small parameter. For the ETE, this parameter is $\tau U/L$, where $U$ is the fluid speed and $L$ is the macroscopic size scale for changes in $U^a$ or $f_0$. The ETE follows from keeping terms linear in $\tau U/L$. The linearized first and second moments of the kinetic equation are as follows.

The first moment:

$$
\frac{d}{dt} \frac{P_f \Lambda_{ab}}{3m} - \frac{m}{dt} \frac{P_f}{3 \partial P^a} \left( pS^a \right) + \frac{q}{mc} \left( \epsilon_{abc} P_{bc} B^a \right) = \frac{-S^a}{\tau}. \tag{4}
$$

The second moment with the trace removed:

$$
- \frac{P_f}{15 \partial P^a} \Lambda_{ab} + \frac{q}{mc} \left[ \epsilon_{abc} P_{bc} B^a + \epsilon_{abc} P_{ad} B^a \right] = \frac{-P_{ab}}{\tau}. \tag{5}
$$

So the first and second moments have been reduced from differential equations to algebraic equations involving $f_0$. The solutions can be written in terms of a diffusion tensor $\kappa_{ab}$ and a viscous tensor $\eta_{abcd}$:

$$
S^a = \kappa_{ab} \left( \frac{m^2 \partial}{\partial P^a} \frac{dU^b}{dt} - \frac{dP_f}{dx^b} \right), \tag{6}
$$

$$
P_{ab} = \eta_{abcd} \frac{m^2 \partial}{\partial P^a} \frac{dU^b}{dt} = \Lambda_{ab} \equiv \eta \eta_{abcd} \Lambda_{cd}. \tag{7}
$$

For magnetic field aligned along the $z$-axis of a Cartesian coordinate system, the components of $\kappa_{ab}$ and $\eta_{abcd}$ are written down by WJ. For an arbitrary coordinate system, these components are written down by Williams (1993). In WJ, the algebra was complicated because it was not appreciated that there is a contribution to the particle streaming flux proportional to the acceleration, as indicated in (6). This led to a mistake in the algebra of WJ. The two terms

$$
- \frac{P_f}{3 \partial P^a} \left( \frac{dU^a}{dt} \frac{dP_f}{dt} \right) + \epsilon_{abc} \partial \frac{\partial}{\partial P^a} \left( \Omega^2 \frac{dU^a}{dt} \frac{dP_f}{dt} \right)
$$

of WJ equation (22) should be replaced with the single term

$$
\frac{1}{m} \frac{\partial}{\partial P^a} \left( \kappa_{ab} \frac{dP_f}{dt} \frac{dU^b}{dt} \right).
$$

Also, the term involving $\eta_{abcd}$ in equation (22) should be multiplied by $\frac{1}{2}$. Thus a corrected version of WJ equation (22) should read as follows:

$$
\frac{df_0}{dt} = \frac{p}{3} \frac{dU^a}{dt} \frac{dP_f}{dt} + \frac{1}{m} \frac{d}{dx^a} \left( \kappa_{ab} \frac{dP_f}{dx^a} \right)
$$

$$
- \frac{1}{m} \frac{d}{dx^a} \left( \kappa_{ab} \frac{dP_f}{dx^a} \frac{dU^b}{dx^a} \right) - \frac{m}{dt} \frac{dU^a}{dt} \frac{\partial}{\partial P^a} \left( pK_{ab} \frac{dP_f}{dx^a} \right)
$$

$$
- \frac{1}{2} K_{ab} \frac{\partial}{\partial P^a} \left( p\eta \eta_{abcd} \Lambda_{cd} \right). \tag{8}
$$

The reasoning of § 3 of that paper which led to WJ (22) is needlessly convoluted. We propose the simpler view outlined above to arrive at the correct ETE. None of the other results or conclusions contained in WJ are affected by this correction. Closely connected work in other papers by Williams and Jokipii did not share these algebraic errors, and so is unaffected.

4. THE EXTENDED TRANSPORT EQUATION AND ITS BROADER APPLICATION

To summarize the present view, the extended transport equation correct to order $\tau U/L$ and order $U/c$ is merely equations (6) and (7) substituted into the zeroth moment (3). Rewriting the zeroth moment slightly:

$$
\frac{df_0}{dt} + U^a \frac{df_0}{dx^a} + \frac{dS^a}{m dx^a} = \frac{dU^a}{dt} \frac{df_0}{dx^a} - \frac{dU^a}{dt} \frac{dP_f}{dx^a} \frac{dS^a}{m dx^a} + \frac{m}{dt} \frac{dU^a}{dt} \frac{dP_f}{dx^a} \frac{dS^a}{m dx^a}.
$$

Augmented with equations (6) and (7), equation (9) constitutes the ETE. The reduction to the usual diffusion-convection equation occurs when the acceleration terms and the viscous term are ignored. If these terms are kept, all but one of the new terms is order $\tau U/L$ smaller than the convection term. The term that is quadratic in the acceleration vector,

$$
\frac{m}{dt} \frac{dU^a}{dt} \frac{dP_f}{dx^a} \frac{dS^a}{m dx^a} \left( m^2 \frac{dP_f}{dx^a} \frac{dU^b}{dx^a} \right),
$$

is order $(\tau U/L)(m^2 U^2/p)$ smaller than the convection term and order $m^2 U^2/p^2$ smaller than the other acceleration terms and the viscous term. If only energetic particles are considered, then this term is dropped. Earl et al. (1988) and WJ dropped this term since these papers considered modifications to the cosmic-ray transport equation, where particles are presumed to have speeds much greater than the flow speed.
This consideration raises the key point of this paper: a simple change in view regarding the ordering parameter allows us to obtain a transport equation free from restrictions on particle speeds. The small parameter $\tau U/L = (\lambda/L)(mU/p)$, where $\lambda = pr/m$ is a scattering mean free path. We can construct a transport equation for particles of low energy by choosing our smallness parameter to be $\lambda/L$. Equations (6), (7) and (8) are now correct to first order in this small parameter and can be applied to particles with all energies. If $p \sim mU$, the term quadratic in the acceleration must be kept because it, along with the other acceleration terms, the viscous term, and the diffusion term are all the same size: of order $\lambda/L$ smaller than the convection term. If $p \ll mU$, then the term quadratic in the acceleration dominates the viscous term and also dominates the other inertial terms, which in turn dominate the diffusion term.

Thus we obtain a transport equation correct to first order in $\lambda/L$ and first order in $U/c$ that applies to particles of arbitrary energy, and has terms which have relative sizes that depend on particle momenta. This equation will be important in the study of the acceleration of thermal particles to high energies. If particle momenta are $mU$, then the viscous and inertial terms are smaller than the diffusion term by a factor of at least $m^2U^2/p^2$, and the equation reduces to Parker’s original diffusion-convection equation.

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REFERENCES

Parker, E. N. 1965, Planet. Space Sci., 13, 9