THE TELEGRAPH EQUATION IN CHARGED PARTICLE TRANSPORT

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ABSTRACT

We present a new derivation of the telegraph equation which modifies its coefficients. First, an infinite-order partial differential equation is obtained for the velocity-space solid angle-averaged phase-space distribution of particles which underwent at least a few collisions. It is shown that in the lowest order asymptotic expansion this equation simplifies to the well-known diffusion equation. The second-order asymptotic expansion for isotropic small-angle scattering results in a modified telegraph equation with a signal propagation speed of \( v(5/11)^{1/2} \) instead of the usual \( v^{3/2} \). Our derivation of a modified telegraph equation follows from an expansion of the Boltzmann equation in the relevant smallness parameters and not from a truncation of an eigenfunction expansion. This equation is consistent with causality. It is shown that under steady state conditions in a convecting plasma the telegraph equation may be regarded as a diffusion equation with a modified transport coefficient, which describes a combination of diffusion and cosmic-ray inertia. This modified transport coefficient becomes negative for particles with random velocities less than the critical velocity, \( v_c \). This negative value is a consequence of the second time derivative term in the telegraph equation and it is closely related to causality.

Subject headings: acceleration of particles — cosmic rays — diffusion

1. INTRODUCTION

For the last quarter of century, energetic particle transport theory has been based on Parker's (1958, 1965) equation originally formulated to describe the solar modulation of Galactic cosmic-ray density. The Parker equation describes the evolution of the velocity-space solid angle-averaged phase-space distribution function due to convection, adiabatic compression, and spatial diffusion. It is assumed that the energetic particles move much faster than the bulk speed of the plasma, \( u \). In this approximation, superthermal particles have no difficulty to diffuse in any direction and they instantaneously reach any spatial point, thus violating causality. In early work, a Fokker-Planck approach was used to obtain the relationship between the spatial diffusion coefficient and magnetic field fluctuations (cf. Jokipii 1966, 1971). Later, quasi-linear theory has been applied to describe particle diffusion due to wave-particle interactions (cf. Jokipii 1971; Kennel & Engleman 1966; Völk 1975; Lee 1982). The problem of infinite signal propagation speed was also recognized and addressed quite early by Axford (1965) using a linear expansion of the distribution function.

Recently, the acceleration of cometary pickup ions presented an interesting challenge to traditional energetic particle transport theory and underlined the lack of causality in the diffusion approximation. At pickup, newly born cometary ions have identical random and bulk speeds. As a result of the interaction with self-generated cometary waves, pickup ions undergo considerable velocity space diffusion (for a detailed discussion and an extensive list of relevant references see the recent review of Gombosi 1991). The velocity space diffusion of these pickup ions also leads to diffusion in configuration space. These particles have random velocities close to that of the plasma bulk speed, therefore the lack of causality leads to obvious contradictions. According to the "classical" diffusion picture particles with \( v \approx u \) could diffuse upstream (against the flow) in the plasma frame. On the other hand, it is clear that in the frame of a stationary observer particles with \( v < u \) cannot diffuse upstream, because they are convected downstream at a velocity exceeding their random velocity. In this paper we present a new transport equation for the velocity-space solid angle-averaged phase-space distribution function which satisfies causality.

Recently Earl, Jokipii, & Morfill (1988) and Williams & Jokipii (1991) used a simple scattering model and took velocity-space solid angle-averaged velocity moments of the Boltzmann equation to derive a new cosmic-ray transport equation which included higher order effects such as cosmic-ray viscosity. In this paper we follow their approach and show that for low-energy particles the transport equation becomes a telegrapher's equation, which describes finite propagation speed for particle diffusion.

2. A SIMPLE TRANSPORT MODEL

Let us consider a background plasma (such as the solar wind) contaminated with a nonthermal energetic particle population (such as cosmic rays or pickup ions). The temporal and spatial variation of the plasma bulk flow velocity function, \( \mathbf{u}(t, \mathbf{r}) \), is determined by an additional set of transport equations: In this paper it is assumed to be a known function of time and location.

Assuming that the background plasma is highly conducting (i.e., the electric field vector, \( \mathbf{E} \), is given by the motional electric field, \( \mathbf{E} = -\mathbf{u} \times \mathbf{B} \), where \( \mathbf{B} \) is the magnetic field vector), the evolution of the superthermal particle distribution function, \( f(t, \mathbf{r}, \mathbf{v}) \), is described by the following Boltzmann equation (the equation is given in a phase-space coordinate system where the configuration space is inertial, but the velocity space frame of
reference is moving together with the plasma; Burgers 1969):  
\[
\frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + v_i \frac{\partial f}{\partial x_i} = \left( \Omega \epsilon_{ijk} v_j b_k - \frac{\partial u_i}{\partial t} - u_j \frac{\partial u_i}{\partial x_j} - v_j \frac{\partial u_i}{\partial x_j} \right) \frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial t} \bigg|_{\text{coll}},
\]
where $x_i$ represent spatial coordinates ($i = 1, 2, 3$), $v_i$ are random velocity components, $\Omega = qB/m$ is the gyrofrequency (its sign depends on the particle charge, $q$), $B_i$ are the components of the magnetic field vector, $b_i = B_i/B$ is the unit vector along the magnetic field, $m = $ particle mass, $\epsilon_{ijk} =$ permutation tensor, and $(\partial f/\partial t)_{\text{coll}} =$ rate of change of the distribution function due to particle production, loss, and collisions (including wave-particle interactions). Repeated indices indicate summation.

For the sake of mathematical simplicity it is assumed that collisions randomize the pitch-angle distribution but do not change the magnitude of the particle velocity. This approximation neglects potentially important processes, such as mass loading or velocity diffusion. These additional effects can be incorporated to the transport equation later. However, at this point we want to concentrate on the main issue of spatial diffusion, and this simplified collision term helps us to emphasize the important physical points. In this limit the relation time approximation is applied for the collision term:
\[
\frac{\partial f}{\partial t} = -\frac{f - f_0}{\tau},
\]
where $f_0(t, r, v) = \langle f(t, r, v) \rangle$ (throughout this paper $\langle \rangle$ denotes velocity-space solid angle–averaged quantities at constant speed, $v$) and $\tau(t, r, v)$ is a formally defined pitch-angle scattering mean free time. The same collision term was used by Earl et al. (1988) and Williams & Jokipii (1991) in their studies of cosmic-ray viscosity. The case of small-angle scattering yields similar results and is discussed below.

First we consider a very simple transport scenario and carry out a velocity-space solid angle moment analysis of the Boltzmann equation. The consideration of this simple situation ensures mathematical simplicity and helps us to concentrate on the important physical points. In subsequent sections we will apply our conclusions to more complicated situations.

Let us consider the transport of charged test particles in a fully ionized, stationary plasma with a strong, uniform magnetic field ($u_i = 0$ everywhere). It is assumed that the charged-particle distribution function, $f$, has no gradient perpendicular to the magnetic field lines. This assumption has two immediate mathematical consequences: in velocity space $f$ is independent of the phase angle of gyration, $\phi$, while in configuration space it depends only on the distance along the magnetic field line, $x$. In summary, one can write that $f = f(t, x, v, \mu)$, where $v$ is the particle speed (with respect of the stationary background plasma) and $\mu$ is the pitch angle. In this simple scenario the Boltzmann equation for these charged-particles can be written as
\[
\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} = -\frac{f(t, x, v, \mu) - f_0(t, x, v)}{\tau}.
\]

3. THE SHORT-TIME LIMIT SOLUTION

We first consider the solution of equation (3) for very short time scales ($t \ll \tau$), when the scattering of particles did not modify the distribution function in any significant way. The propagation of particles for a specified initial distribution function $g_0(x, v, \mu)$, for times $t \ll \tau$, much smaller than a scattering time, $\tau$, is described by the following scatter-free transport equation:
\[
\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} = 0.
\]

The general solution to equation (4) may be written as
\[
f(t, x, v, \mu) = g_0(x - \mu vt, v, \mu),
\]
which simply states that the guiding centers of the particles travel along straight lines.

If we consider the simple case of an isotropic distribution containing $N$ particles instantaneously injected at $x = 0$, $t = 0$, and $v = v_0$, the solution of equation (5) becomes
\[
f(t, x, v, \mu) = \frac{N}{4\pi v_0^2} \delta(x - \mu vt)\delta(v - v_0).
\]

From this result we can immediately calculate the number density as a function of time and distance:
\[
N(t, x) = \iiint d^3v f(t, x, v, \mu) = \frac{N}{2\pi^2} H(vt + x)H(vt - x),
\]
where $H(x)$ is the step function. This solution is a rectangle which begins at $x = -vt$ and extends to $x = +vt$. This solution, which follows exactly from Liouville's equation, is to be contrasted with collisional solutions discussed below. It should be noted that at a given spatial location this solution loses validity for times $t > x/v$.

It should be pointed out that whereas the number density is a step function in time (see eq. [7]), the distribution function at any given pitch angle has a pulselike behavior. An observer looking in a fixed direction with respect of the magnetic field will see all unscattered particles at the same time. The height of these pulses decays exponentially with time due to scattering. For small pitch angles (nearly field-aligned propagation), the decaying pulse of scatter-free particles remains a prominent feature. For particles with $\sim 90\degree$ pitch angle the scatter-free propagation time is quite long; therefore the observer sees a highly eroded pulse almost buried in the background of scattered particles. These effects have been discussed previously by Fisk & Axford (1969) and Earl (1974).

4. MOMENT ANALYSIS OF THE TRANSPORT EQUATION

In what follows we will carry out a moment analysis of the collisional transport equation (eq. [3]) with respect of the particle pitch angle, $\mu$. As a first step we expand both the distribution function and the collision term into an infinite series of orthogonal polynomials. Using Legendre polynomials as the orthogonal polynomial series the two functions can be written in the following form:
\[
f(t, x, v, \mu) = \sum_{n=0}^{\infty} f_n(t, x, v) P_n(\mu),
\]
\[
\frac{\partial f}{\partial t} \bigg|_{\text{coll}} = -\frac{1}{\tau} \sum_{n=1}^{\infty} f_n(t, x, v) P_n(\mu).
\]

Here $f_n$ is the $n$th harmonics of the distribution function, $P_n$ represents the $n$th Legendre polynomial, and $\tau$ is a formally
defined time constant (assumed to be independent of $x$ and $t$) which describes the isotropization of $f$ due to collisions. This simple form of the collision term assumes isotropic collisions. Substituting the Legendre polynomial expansions into the Boltzmann equation results in the following differential equation:

$$\frac{\partial f_0}{\partial t} + v \frac{\partial f_0}{\partial x} = -\frac{1}{\tau} \sum_{n=1}^{\infty} f_n(t, x, v) P_n(\mu).$$

Next we multiply this equation by $P_n(\mu)$ and integrate the result from $-1$ to $+1$. This operation simplifies equation (9) to the following (here we used the orthogonality relation of Legendre polynomials, i.e.,

$$(2k + 1) \int_{-1}^{1} d\mu \mu P_k(\mu) P_n(\mu) = 2\delta_{kn},$$

$$\frac{2}{2k + 1} (1 + \tau \frac{\partial}{\partial t}) f_k = -\frac{k}{2k - 1} \frac{\partial f_0}{\partial x} - \frac{k + 1}{2k + 3} \frac{\partial f_{k+1}}{\partial x},$$

for $k$ values larger than 0. It should be noted that for $k = 0$ we get

$$\frac{2}{2k + 1} \tau \frac{\partial f_0}{\partial t} = -\frac{\partial f_0}{\partial x} = \frac{2}{2k + 1} \frac{\partial}{\partial t} \left( -\frac{\partial f_0}{\partial x} \right).$$

(10a)

The integral in the right-hand side of equations (10) and (10a) can be evaluated by using the following recursive formula:

$$\mu P_k = \frac{(k + 1)P_{k+1} + kP_{k-1}}{2k + 1}.$$

Now for $k > 0$ values equation (10) can be written as

$$\left( 1 + \tau \frac{\partial}{\partial t} \right) f_k = -\frac{k}{2k - 1} \frac{\partial f_0}{\partial x} - \frac{k + 1}{2k + 3} \frac{\partial f_{k+1}}{\partial x},$$

where $\lambda = v\tau$. For $k = 0$ equation (10a) becomes

$$\tau \frac{\partial f_0}{\partial t} = -\frac{1}{3} \frac{\partial f_1}{\partial x}. $$

(13a)

Specifically, for $k = 1$ through 5 values equation (12) becomes

$$\left( 1 + \tau \frac{\partial}{\partial t} \right) f_1 = -\frac{\lambda}{5} \frac{\partial f_1}{\partial x} - \frac{2}{5} \frac{\partial f_2}{\partial x},$$

(13b)

$$\left( 1 + \tau \frac{\partial}{\partial t} \right) f_2 = -\frac{2}{3} \frac{\partial f_2}{\partial x} - \frac{3}{7} \frac{\partial f_3}{\partial x},$$

(13c)

$$\left( 1 + \tau \frac{\partial}{\partial t} \right) f_3 = -\frac{3}{5} \frac{\partial f_3}{\partial x} - \frac{4}{9} \frac{\partial f_4}{\partial x},$$

(13d)

$$\left( 1 + \tau \frac{\partial}{\partial t} \right) f_4 = -\frac{4}{7} \frac{\partial f_4}{\partial x} - \frac{5}{9} \frac{\partial f_5}{\partial x},$$

(13e)

$$\left( 1 + \tau \frac{\partial}{\partial t} \right) f_5 = -\frac{5}{9} \frac{\partial f_5}{\partial x} - \frac{6}{13} \frac{\partial f_6}{\partial x}. $$

(13f)

Note that equation (12) represents an infinite number of first-order differential equations for the harmonics of the distribution function $f_0$.

5. TRANSPORT EQUATION FOR $f_0$

In this section the infinite number of first-order partial differential equations given by expression (12) will be transformed to a single partial differential equation for the velocity-space solid angle-averaged distribution function $f_0$.

The first step is to apply the $\frac{\partial}{\partial x}$ operator to equation (12). This operation results in the following equation:

$$\left( 1 + \tau \frac{\partial}{\partial t} \right) \frac{\partial f_0}{\partial x} = \frac{k}{2k - 1} \frac{\partial f_{k+1}}{\partial x} - \frac{k + 1}{2k + 3} \frac{\partial f_{k+1}}{\partial x}. $$

(14)

In the second step we take equation (13) connecting the zeroth and the first harmonics and substitute it to equation (14) for $k = 1$. After some manipulations one obtains

$$\frac{2}{15} \lambda \frac{v}{\tau} \frac{\partial f_2}{\partial x} = \left( 1 + \tau \frac{\partial}{\partial t} \right) \frac{\partial f_0}{\partial x} - \frac{1}{2} \frac{\partial f_0}{\partial x}. $$

(15)

Equation (15) is a second-order partial differential equation for $f_0$. If the directional distribution is close to isotropy the second harmonics of the distribution function can be neglected; in other words we can take $f_2 = 0$. In this case we get the following transport equation:

$$\left( 1 + \tau \frac{\partial}{\partial t} \right) \frac{\partial f_0}{\partial x} - \frac{\lambda \frac{v}{\tau} \frac{\partial^2 f_0}{\partial x^2}}{3} = 0. $$

(16)

Equation (16) is the standard telegrapher's equation with a “signal” propagation speed of $v/\sqrt{3}$ (cf. Axford 1965). For an isotropic pitch angle distribution $v/\sqrt{3}$ is the average particle velocity component along the magnetic field. Equation (16) is valid only in situations when the distribution is isotropic enough at every spatial point and there is equipartition between the various components of kinetic energy. However, we will see below that simply truncating the series is not consistent with a physically realizable ordering of smallness parameters. We must carry out a consistent asymptotic expansion.

Next we take equation (14) with $k = 2$ and substitute equations (13a) and (15):

$$\frac{6}{105} \frac{v^2}{\tau} \frac{\partial^3 f_0}{\partial x^3} = \left( 1 + 2\tau \frac{\partial}{\partial t} + \tau^2 \frac{\partial^2}{\partial t^2} \right) \frac{\partial f_0}{\partial t} - \left( 1 + \frac{9}{5} \tau \frac{\partial}{\partial t} \right) \frac{\lambda \frac{v}{\tau} \frac{\partial^2 f_0}{\partial x^2}}{3} = 0. $$

(17)

If we stop our Legendre polynomial expansion after the second harmonics ($f_2 = 0$), equation (17) results in the following equation for $f_0$:

$$\left( 1 + 2\tau \frac{\partial}{\partial t} + \tau^2 \frac{\partial^2}{\partial t^2} \right) \frac{\partial f_0}{\partial t} - \left( 1 + \frac{9}{5} \tau \frac{\partial}{\partial t} \right) \frac{\lambda \frac{v}{\tau} \frac{\partial^2 f_0}{\partial x^2}}{3} = 0. $$

(18)

The same procedure can be continued with truncation at the fourth and fifth harmonics. These approximations lead to the increasingly complicated equations for $f_0$. For $f_4 = 0$ we obtain

$$\left( 1 + 3\tau \frac{\partial}{\partial t} + \tau^2 \frac{\partial^2}{\partial t^2} + \tau^3 \frac{\partial^3}{\partial t^3} \right) \frac{\partial f_0}{\partial t} - \left( 1 + \frac{25}{7} \tau \frac{\partial}{\partial t} + \frac{18}{7} \tau^2 \frac{\partial^2}{\partial t^2} + \frac{3}{35} \tau^3 \frac{\partial^3}{\partial t^3} \right) \frac{\lambda \frac{v}{\tau} \frac{\partial^2 f_0}{\partial x^2}}{3} = 0. $$

(19)
Truncating at the fifth harmonics ($f_5 = 0$) yields the following equation:

\[
\left(1 + 4 r \frac{\partial}{\partial t} + 6 r^2 \frac{\partial^2}{\partial t^2} + 4 r^3 \frac{\partial^3}{\partial t^3} + r^4 \frac{\partial^4}{\partial t^4}\right) \frac{\partial f_0}{\partial t} - \left(1 + \frac{16}{3} r \frac{\partial}{\partial t} + \frac{23}{3} r^2 \frac{\partial^2}{\partial t^2} + \frac{10}{3} r^3 \frac{\partial^3}{\partial t^3}\right) \frac{\lambda v}{3} \frac{\partial^2 f_0}{\partial x^2} + \left(\frac{23}{45} + \frac{5}{7} r \frac{\partial}{\partial t}\right) \frac{\lambda^2 v}{3} \frac{\partial^4 f_0}{\partial x^4} = 0 .
\]

We continue this elimination process with the higher harmonics and obtain the following general equation for the velocity-space solid angle–averaged phase-space distribution function, $f_0$:

\[
\left[\left(1 + \tau \frac{\partial}{\partial t}\right)^n - \left(1 + \tau \frac{\partial}{\partial t}\right)^{n-2} \left(\sum_{k=2}^{n} A_k \lambda^2 \frac{\partial^2}{\partial x^2}\right) + \left(1 + \tau \frac{\partial}{\partial t}\right)^{n-4} A_2 \left(\sum_{k=3}^{n} A_k \lambda^2 \frac{\partial^4}{\partial x^4} - \cdots\right) \frac{\lambda v}{3} \frac{\partial^2}{\partial x^2} f_0\right] = \left[\left(1 + \tau \frac{\partial}{\partial t}\right)^{n-1} - \left(1 + \tau \frac{\partial}{\partial t}\right)^{n-3} \left(\sum_{k=3}^{n} A_k \lambda^2 \frac{\partial^2}{\partial x^2}\right) + \cdots\right] \frac{\lambda v}{3} \frac{\partial^2}{\partial x^2} f_0 ,\]

where $A_k = k^2/(4k^2 - 1)$, and the series is truncated at the $n$th harmonics ($f_{n+1} = 0$).

Equation (21) represents a single partial differential equation for the velocity-space solid angle–averaged distribution function, $f_0$. Mathematically it is equivalent with the series of first-order partial differential equations described by equation (12).

We are now in a position to consider the telegraph equation (16) which results from truncation at $f_5$. First we examine the limit in which $\tau \partial^2 f_0/\partial t^2 \ll \partial f_0/\partial t$. Then the solution approximately satisfies the diffusion equation, $\partial f_0/\partial t \approx (\lambda t/3 \partial^2 f_0/\partial x^2)$, and the telegraph equation adds $\tau \partial^2 f_0/\partial t^2$. On the other hand, we see from equation (18) that if instead we stop at $f_3$ (by setting $f_5$ to zero), the transport equation contains a term of $2 \tau \partial^2 f_0/\partial t^2$ instead the previous $\tau \partial^2 f_0/\partial t^2$. In general, truncating at $n$th order brings in a term of $(n - 1) \tau \partial^2 f_0/\partial t^2$. All these terms are of the same order in a smallness parameter, but they differ in a numerical factor. Another way of seeing this is to note that if the term $\tau \partial^2 f_0/\partial t^2$ is large enough to produce a large deviation from the diffusion solution, we must have $\tau \partial f_0/\partial t \approx f$, and the higher order harmonics will not be negligible. Instead, the approximation breaks down in a fundamental way. It is clear that simply truncating the solution at some harmonic is not physically correct, and the telegraph equation so obtained is not useful in any realizable situation because it is not correct to a consistent order in a smallness parameter. This difficulty is not specific to the specific scattering operator used and is shown below to extend to small-angle scattering as well.

In the following section we find an asymptotic expansion, which is correct to a given order in a physically specified smallness parameter. In the process we find a "modified" telegraph equation.

6. ASYMPTOTIC EXPANSION

Equations (16), (18), (19), and (20) represent a series of transport equations for the velocity-space solid angle–averaged phase-space distribution function, $f_0$. These equations were obtained using the original Legendre polynomial expansion of the distribution function (given in eq. [8]) and by truncating the expansion after the first, second, third, and fourth harmonics, respectively. Equation (21) represents the general form of these equations, obtained by truncating the series expansion after the $n$th harmonics ($f_{n+1} = 0$).

In this section we examine equation (21) for time scales larger than a few times the scattering time, $\tau$. It can be seen from equation (9) that in our simple scattering model all but the lowest order harmonics "decay" with a characteristic time scale of $\tau$. This means that as time goes on the velocity distribution is becoming more and more isotropic. In equation (21) the evolution toward isotropy is ensured by the differential operators $\tau \partial/\partial t$ and $\lambda^2 \partial^2/\partial x^2$ acting on the solid angle–averaged distribution function, $f_0$. It is reasonable to assume that as time increases $\tau \partial f_0/\partial t \to 0$ and $\lambda^2 \partial^2 f_0/\partial x^2 \to 0$. In this section we consider times when the following two relations are satisfied:

\[
\frac{1}{f_0} \left| \frac{\partial f_0}{\partial t} \right| = |\epsilon_1| \ll 1 ,
\]

\[
\frac{1}{f_0} \left| \frac{\lambda^2 \partial^2 f_0}{\partial x^2} \right| = |\epsilon^2_1| \ll 1 .
\]

Keeping only the leading terms in the temporal and spatial smallness parameters, $\epsilon_1$ and $\epsilon^2_1$, one can write equations (16) through (20) in the following form:

\[
(1 + \epsilon_1) \frac{\partial f_0}{\partial t} - \frac{\lambda v}{3} \frac{\partial^2 f_0}{\partial x^2} = 0 ,
\]

\[
(1 + 2 \epsilon_1) \frac{\partial f_0}{\partial t} - \left(1 + \frac{9}{5} \epsilon_1\right) \frac{\lambda v}{3} \frac{\partial^2 f_0}{\partial x^2} = 0 ,
\]

\[
(1 + 3 \epsilon_1) \frac{\partial f_0}{\partial t} - \left(1 + \frac{25}{7} \epsilon_1 - \frac{9}{35} \epsilon^2_1\right) \frac{\lambda v}{3} \frac{\partial^2 f_0}{\partial x^2} = 0 ,
\]

\[
(1 + 4 \epsilon_1) \frac{\partial f_0}{\partial t} - \left(1 + \frac{16}{3} \epsilon_1 - \frac{11}{21} \epsilon^2_1\right) \frac{\lambda v}{3} \frac{\partial^2 f_0}{\partial x^2} = 0 .
\]

Equations (23a)–(23d) represent an increasingly higher order approximation of the transport equation. They also present us with a dilemma, because the coefficients of $\epsilon_1$ and $\epsilon^2_1$ are different in each equation.

This problem can be resolved by finding a relationship between $\epsilon_1$ and $\epsilon^2_1$. We begin this by writing equation (21) in the following form:

\[
\left[\left(1 + \epsilon_1\right)^n - \left(1 + \epsilon_1\right)^{n-2} \left(\sum_{k=2}^{n} A_k \epsilon^2_1\right)\right] \epsilon_1 + \left(1 + \epsilon_1\right)^{n-4} A_2 \left(\sum_{k=3}^{n} A_k \epsilon^2_1\right) + \cdots = \left[\left(1 + \epsilon_1\right)^{n-1} - \left(1 + \epsilon_1\right)^{n-3} \left(\sum_{k=3}^{n} A_k \epsilon^2_1\right) + \cdots\right] \frac{1}{3} \epsilon^2_1 .
\]

In the lowest order approximation this equation yields

\[
\epsilon_1 = \frac{1}{2} \epsilon^2_1 .
\]

Equation (25) expresses the relationship between the smallest parameters. This equation is accurate to the order of $\epsilon^2_1$. 

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Using equation (25) we can evaluate equation (21) to an arbitrary level of accuracy. In the lowest order approximation we get the well-known diffusion equation:

$$\frac{\partial f_0}{\partial t} - \frac{\lambda v}{3} \frac{\partial^2 f_0}{\partial x^2} = 0.$$  \hspace{1cm} (26)

It should be noted that the "signal" speed described by this diffusion equation is infinitely large. If we have a spatially localized particle distribution at $t = 0$, the solution of equation (26) yields $f_0(t', x) > 0$ for any value of $x$ if $t' > 0$.

In the second-order asymptotic expansion we keep all terms up to $\epsilon^2$ (or $\epsilon^1$):

$$\left[ 1 + nt \frac{\partial}{\partial t} - \left( \sum_{k=3}^{\infty} A_k \right) \frac{\partial^2}{\partial x^2} \right] \left( \frac{\partial f_0}{\partial t} - \frac{\lambda v}{3} \frac{\partial^2 f_0}{\partial x^2} \right) + \left( \frac{1}{3} - A_2 \right) \frac{\partial^2}{\partial x^2} \frac{\partial f_0}{\partial t} = 0. \hspace{1cm} (27)$$

On the other hand, we know that

$$\frac{\partial f_0}{\partial t} - \frac{\lambda v}{3} \frac{\partial^2 f_0}{\partial x^2} = O(\epsilon^1). \hspace{1cm} (28)$$

This means that in our second-order accurate approximation the relevant contribution of square bracket in equation (27) reduces to 1, and the transport equation becomes

$$\frac{\partial f_0}{\partial t} + \frac{\partial^2 f_0}{\partial x^2} + \frac{\lambda v}{3} \frac{\partial^2 f_0}{\partial x^2} = 0. \hspace{1cm} (29)$$

This equation is a telegraph equation describing the evolution of the velocity-space solid angle-averaged phase-space distribution function, $f_0$. Equation (29) is accurate to second order in the smallness parameter, $\epsilon$.

The signal speed, $c_s = \sqrt{v/3}/1^{1/2} \approx 1.3v$, in equation (29) is larger than the particle velocity, $v$. This is not of concern here since we are perturbing the diffusion approximation, which has infinite propagation speed.

Different scattering models result in similar conclusions, with somewhat different coefficients in the second-order asymptotic limit. For instance, in the case of isotropic small-angle scattering our asymptotic expansion results in the following telegrapher’s equation (the detailed derivation is given in the Appendix):

$$\left( \frac{\partial}{\partial t} + \frac{11}{15} \tau_s \frac{\partial^2}{\partial x^2} - \frac{1}{3} \tau_s \frac{\partial^2}{\partial x^2} \right) f_0 = 0,$$  \hspace{1cm} (30)

where $\tau_s$ is the characteristic time scale of small-angle scattering. Equation (30) yields a signal speed of $v/3^{3/12} \approx 0.67v$ as opposed to the $v/3^{3/12}$ which follows from simple truncation.

It is worth mentioning that equation (29) (or 30) can also be obtained by truncating at the third harmonic in equation (13c), omitting the time derivative of the second harmonic and using the diffusive approximation to combine the highest derivatives. This procedure explicitly shows that it is physically incorrect to truncate the series at the second harmonic, since the contribution of the second harmonic affects the first two. At the same time this procedure cannot address the potential contribution of even higher harmonics to the telegrapher’s equation. Our asymptotic expansion has shown that the third and higher harmonics do not have any additional contribution to the modified telegraph equation.

7. DISCUSSION

We have presented a new derivation of the telegraph equation for two isotropic scattering operators. Our derivation proceeds via a consistent asymptotic expansion in $\epsilon = (v/3)^{1/2}$, where $r$ is the scattering time. This results in "modified" telegraph equations which have somewhat different coefficients of the second time derivative for the different scattering operators. The analysis suggests that previous derivations (e.g., Axford 1965; Earl 1974) which simply truncate the eigenfunction expansion after the second term may not be consistent to a given order in $\epsilon$. We note that Fisk & Axford (1969) considered a simple "two-stream" model in which $\mu$ was constrained to be $+1$ or $-1$ only. In this case the telegraph equation is correct to all orders. The second time derivative term is connected with the inertia of the energetic particle gas. The usual telegraph equation is equivalent to a two-stream model where forward and backward moving particles have speeds of $+v/3^{1/2}$ and $-v/3^{1/2}$, respectively. This is not surprising, since the derivation relies on truncation at the second moment (i.e., neglecting the third and higher moments). This restriction of the pitch-angle distribution keeps the quadrupole moment zero at all times. The modifications suggested in the presented work are related to the presence of a nonzero quadrupole moment. It is pointed out that the contribution of the nonzero quadrupole moment is comparable to the magnitude of the inertial term.

The modified telegraph equation derived in the previous section for $f_0(t, x)$ is valid for times $t > (r + x)/v$. It should be emphasized that the basic assumptions used in the derivation of equation (30) lose their validity for short times and therefore this equation is not applicable to describe the first (and unscattered) particles arriving to any particular spatial location. Equation (30), however, gives a good description of those particles which underwent a few collisions before arriving to the particular location.

Closely analogous to the time-evolution problem is the spatial distribution of particles injected into a moving fluid. This point can be illustrated by a simple example. Let us transform equation (30) to a coordinate system moving with a non-relativistic constant velocity, $u$, along the $x$ axis. In this case equation (30) can be written as

$$\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) f_0 + \frac{11}{15} \tau_s \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) f_0 - \frac{\tau_s v^2}{3} \frac{\partial^2 f_0}{\partial x^2} = q + \tau_s \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) q,$$  \hspace{1cm} (31)

where $q(t, x)$ represents the source term which is assumed to be isotropic in the moving frame (nonisotropic injection could also be handled without difficulty). It should be mentioned that the numerical factor multiplying the second derivative is a function of the scattering model. In the present case it is $11/5$, for hard sphere collisions it becomes $3/5$, while for the standard telegraph equation (bidirectional propagation) it is 1. We point out that equation (31) is not quantitatively accurate close to the source, where the anisotropy becomes very large and the neglected higher order terms become important.

In order to illustrate the physics let us consider the one-dimensional two-stream model which can be solved analyti-
cally. Assuming a steady-state situation and a continuous point source at \( x = 0 \) equation (31) simplifies to the following equation everywhere except at \( x = 0 \):

\[
\frac{\partial}{\partial x} \left[ u f_0 - \tau_s v^2 \left( 1 - \frac{u^2}{v^2} \right) \frac{\partial f_0}{\partial x} \right] = 0 .
\] (32)

Equation (32) has two solutions:

\[
f_0 = \text{const} , \quad f_0 = A \exp \left( \frac{ux}{\kappa} \right) ,
\] (33)

where \( A \) is a constant and we introduced the quantity

\[
\kappa = \tau_s v^2 \left( 1 - \frac{u^2}{v^2} \right) .
\] (34)

The two solutions are valid for \( x > 0 \) and \( x < 0 \) and they must be matched at the source.

Equation (34) shows an interesting feature: the coefficient, \( \kappa \), becomes negative for random velocities \( v < u \). It should be recognized that \( \kappa \) is indeed a combination of diffusion and inertia, therefore the negative value of \( \kappa \) does not imply negative diffusion. It merely implies that at time scales shorter than the scattering time, \( \tau_s \), particle transport is dominated by unscattered inertial movement. The negative value of \( \kappa \) is a consequence of the second time derivative term in the telegraph equation and it is closely related to causality. In order to illustrate this point we find the solution of the steady state equation (32) in the case of a steady, continuous particle source at \( x = 0 \). This problem has two different solutions depending on the sign of \( \kappa \). For positive values of \( \kappa \), that is for \( u < v \), the solution is

\[
f_0(x) = \begin{cases} 
\frac{Q}{u} \frac{v^2}{v^2 - u^2} \exp \left( \frac{ux}{\kappa} \right) , & x < 0 ; \\
\frac{Q}{u} , & x > 0 
\end{cases}
\] (35)

where \( Q \) is normalization constant characterizing the particle source. For \( u > v \) (i.e., \( \kappa < 0 \)) the solution becomes

\[
f_0(x) = \begin{cases} 
0 , & x < 0 ; \\
\frac{Q}{u} \left[ 1 - \frac{v^2}{v^2 - u^2} \exp \left( \frac{ux}{\kappa} \right) \right] , & x > 0 .
\end{cases}
\] (36)

These two distinct solutions are shown in Figure 1. Solid and dashed lines represent solutions for \( \kappa < 0 \) (\( u > v \)) and \( \kappa > 0 \) (\( u < v \)), respectively. As the analytic solution and Figure 1 indicate \( f_0 \) may change discontinuously at \( x = 0 \). This cannot happen in the diffusion limit.

The picture remains qualitatively the same for the solutions of equation (31) with

\[
\kappa = \tau_s v^2 \left( 1 - \frac{u^2}{5 v^2} \right) .
\] (37)

In this more general case one cannot expect a quantitatively very accurate solution since the population close to the source is dominated by newly produced unscattered particles. Nevertheless, equation (31) qualitatively correctly reflects the fact that the particles cannot diffuse upstream if the fluid velocity happens to be larger than the particle velocity.

\[
\text{FIG. 1.—Steady state velocity-space solid angle–averaged phase-space distribution (in arbitrary units) in a convective plasma in terms of the dimensionless variable, } \chi = ux/\kappa . \text{ Solid and dashed lines represent solutions for } \kappa < 0 \text{ and } \kappa > 0 , \text{ respectively.}
\]

8. SUMMARY

This paper explored causality in charged-particle transport using a simple model. In the short-time limit a scatter-free solution was obtained, which properly reflects causality. An infinite order partial differential equation was obtained for the velocity-space solid angle–averaged phase-space distribution of particles which underwent at least a few collisions. We demonstrated that simply truncating a spherical harmonic expansion produces an equation which is not correctly ordered. A proper asymptotic expansion is derived. It was shown that in the lowest order this simplifies to the well-known diffusion equation. The second-order asymptotic expansion results in a modified telegraph equation with a signal propagation speed of \( v(5/11)^{1/2} \) (assuming isotropic scattering). Our derivation of a modified telegraph equation follows from an expansion of the Boltzmann equation in the relevant smallness parameters. This equation is fully consistent with causality.

It was shown with the help of our simple transport model that under steady state conditions in a convecting plasma the telegraph equation becomes a diffusion equation with a modified spatial diffusion coefficient. This modified diffusion coefficient becomes negative for particles with random velocities less than the critical velocity, \( v_r \). It was shown that the negative diffusion coefficient indicates the inability of these particles to diffuse upstream (against the flow).

Our new telegraph equation is more accurate than the standard one which it replaces, because it keeps all terms up to second order in the smallest parameter, \( \epsilon \). However, for some scattering laws it gives a "signal speed" which is larger than the random particle speed, which is unphysical. This is not of concern since, under circumstances where there would be a contradiction, the approximation breaks down and higher order terms would be required to give an accurate result.

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TELEGRAPH EQUATION FOR ISOTROPIC SMALL-ANGLE SCATTERING

In the case of small-angle isotropic scattering the collision term in equation (3) becomes the following (cf. Jokipii 1971):

$$
\frac{\delta f}{\delta t}_{\text{coll}} = \frac{1}{2 \tau_*} \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial f}{\partial \mu} \right],
$$

(A1)

where the characteristic scattering time, $\tau_*$, is independent of $\mu$. Substituting the Legendre polynomial expansion for the distribution function (cf. eq. [8]) yields the form of this collision term

$$
\frac{1}{\tau_*} \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \sum_{n=0}^{\infty} \frac{\partial f_n(t, x, v)}{\partial \mu} P_n(\mu) \right] = \sum_{n=0}^{\infty} \frac{f_n(t, x, v)}{\tau_n} P_n(\mu),
$$

(A2)

where we introduced the time constant of the $n$th harmonics, $\tau_n$, by the following definition. For $n = 0$ the time constant is zero and for $n > 0$ values it is given by

$$
\tau_n = \frac{2 \tau_*}{n(n + 1)}.
$$

(A3)

Next we substitute this collision term into equation (9) and obtain the following form of the simplified Boltzmann equation:

$$
\sum_{n=0}^{\infty} \left( \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x} + \frac{1}{\tau_n} \right) f_n(t, x, v) P_n(\mu) = 0.
$$

(A4)

Next we multiply this equation by $P_n(\mu)$ and integrate the result from $\mu = -1$ to $\mu = +1$. This operation yields an infinite series of coupled first-order equations for $f_k$. For $k = 0$ we get an identical equation to equation (13a):

$$
\frac{1}{3} \lambda \frac{\partial f_k}{\partial x} = -\tau_* \frac{\partial f_0}{\partial t},
$$

(A5)

where $\lambda = v \tau_*$. For $k > 0$ the equation is the following:

$$
\frac{2}{k(2k + 3)} \lambda \frac{\partial^2 f_{k+1}}{\partial x^2} = -D_k f_k - \frac{2}{(k + 1)(2k - 1)} \lambda \frac{\partial f_{k-1}}{\partial x},
$$

(A6)

where

$$
D_k = \left( 1 + \tau_k \frac{\partial}{\partial t} \right)
$$

(A7)

for $k > 0$. For $k = 0$ we define the differential operator as $D_0 = \tau_* \partial / \partial t$.

The infinite number of first-order differential equations can be transformed to a single higher order partial differential equation for $f_0$. We start by applying the $\partial^2 / \partial x^2$ operator to equation (A6):

$$
\frac{2}{k(2k + 3)} \lambda \frac{\partial^{k+1} f_{k+1}}{\partial x^{k+1}} = -D_k \frac{\partial f_k}{\partial x^k} - \frac{2}{(k + 1)(2k - 1)} \lambda \frac{\partial^{k+1} f_{k-1}}{\partial x^{k+1}}.
$$

(A8)

We take the $k = 1, 2, 3,$ and 4 cases and use equations (A4) and (A7) to obtain

$$
\frac{\partial f_2}{\partial x^2} = \frac{b_1}{\lambda^2} (D_0 D_1 - a_1 \Delta_x) f_0,
$$

(A9a)

$$
\frac{\partial f_3}{\partial x^3} = \frac{b_2}{\lambda^3} [D_0 D_1 D_2 - (a_2 D_0 + a_1 D_1) \Delta_x] f_0,
$$

(A9b)

$$
\frac{\partial f_4}{\partial x^4} = \frac{b_3}{\lambda^4} [D_0 D_1 D_2 D_3 - (a_3 D_0 D_1 + a_2 D_0 D_3 + a_1 D_1 D_3) \Delta_x + a_1 a_3 \Delta_x^2] f_0,
$$

(A9c)

$$
\frac{\partial f_5}{\partial x^5} = \frac{b_4}{\lambda^5} [D_0 D_1 D_2 D_3 D_4 - (a_3 D_0 D_1 D_4 + a_2 D_0 D_3 D_4 + a_1 D_1 D_3 D_4 + a_4 D_0 D_1 D_4) \Delta_x + (a_2 a_4 D_0 + a_1 a_4 D_2 + a_1 a_3 D_4) \Delta_x^2] f_0,
$$

(A9d)
where the operator, $\Delta_x$, and the coefficients, $a_k$, and $b_k$, are defined as follows:

$$
\Delta_x = \lambda^2 \frac{\partial}{\partial x^2} \tag{A10a}
$$

$$
a_k = \begin{cases} 
\frac{1}{3}, & k = 1; \\
\frac{4}{(k^2 - 1)(4k^2 - 1)}, & k > 1; 
\end{cases} \tag{A10b}
$$

$$
b_k = (-1)^{k+1} \frac{k! (2k + 3)(2k + 1) \cdots 1}{2^k} \tag{A10c}
$$

If we truncate our series after the $k$th harmonics, i.e., by assuming that $f_{k+1} = 0$, the following $k$th order partial differential equation is obtained for the solid angle–averaged distribution function, $f_0$:

$$
\left[ D_0 \prod_{l=1}^{k} D_k - a_1 \Delta_x \prod_{l=2}^{k} D_k - D_0 \Delta_x \sum_{l=2}^{k} a_l \left( \prod_{j=1}^{l-2} D_j \prod_{n=l+1}^{k} D_n \right) + a_1 \Delta_x^2 \sum_{l=3}^{k} a_l \left( \prod_{j=2}^{l-2} D_j \prod_{n=l+1}^{k} D_n \right) + \cdots \right] f_0 = 0. \tag{A11}
$$

Next we will carry out the same asymptotic expansion discussed in the main body of the paper. In the first order we get the well-known diffusion equation

$$
(D_0 - a_1 \Delta_x) f_0 = 0. \tag{A12a}
$$

This equation can also be written in an explicit form

$$
\left( \tau_0 \frac{\partial}{\partial t} - \frac{1}{3} \lambda \tau_0 \frac{\partial^2}{\partial x^2} \right) f_0 = 0. \tag{A12b}
$$

The second-order asymptotic expansion in the smallness parameter yields the following equation:

$$
\left[ D_0 + D_0^2 - a_1 \Delta_x - \Delta_x D_0 a_2 + \left( D_0 \sum_{l=2}^{k} \frac{\tau_l}{\sum_{l=3}^{k} a_l} (D_0 - a_1 \Delta_x) \right) \right] f_0 = 0. \tag{A13}
$$

On the other hand, we know that $(D_0 - a_1 \Delta_x)$ is the second order in the smallness parameter, and when this difference is multiplied by $D_0$ or $\Delta_x$ the result is higher order. This means that equation (A16) simplifies to the following:

$$
(D_0 + D_0^2 - a_1 \Delta_x - a_1 \Delta_x) f_0 = 0. \tag{A14}
$$

Substituting the appropriate constants we obtain the following telegrapher’s equation:

$$
\left( \frac{\partial}{\partial t} + \frac{11}{15} \frac{\tau_0}{\tau_*} \frac{\partial^2}{\partial x^2} - \frac{1}{3} \tau_* v^2 \frac{\partial^2}{\partial x^2} \right) f_0 = 0. \tag{A15}
$$

This equation has a signal speed of $c(5/11)^{1/2}$.

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