Analysis of the Ionosphere-Plasmasphere Transport of Superthermal Electrons

1. Transport in the Plasmasphere

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Analytic solutions to the kinetic equation which describes the transport of superthermal electrons in the plasmasphere are presented. Relations which allow the calculation of the photoelectron energy spectra and the plasmaspheric transparency are obtained. An expression for the heating rate of the thermal electron population by the superthermal electrons is also given.

1. INTRODUCTION

Superthermal electrons created in the upper atmosphere by either solar EUV radiation (photoelectrons) or by impact ionization play an important role in the ionosphere and plasmasphere. These electrons heat the thermal plasma, create further ionization, excite atomic and molecular species, and generate plasma oscillations. The complex structure of the superthermal electron source and the many processes affecting these electrons make an exact formulation and solution of the kinetic equation describing their behavior very difficult. A great deal of effort has been devoted to this problem during the last couple of decades, and a number of different approaches have been employed to get meaningful and in general numerical solutions to this problem (for a comprehensive review see Khazanov et al., manuscript in preparation, 1992). There are still significant uncertainties associated with many of the collision cross sections which are necessary for accurate quantitative calculations, therefore the present numerical schemes appear to be sufficient, in general. One aspect of the superthermal electron problem is their transport through the plasmasphere. This issue has been looked at in the past, but no satisfactory and/or comprehensive solutions have been obtained so far [Hanson, 1963; Gastman, 1973; Takahashi, 1973; Lejeune and Wormser, 1976].

The Boltzmann kinetic equation for superthermal electrons at altitudes greater than about 1000 km can be written as [Krinsberg, 1978; Khazanov, 1979]

\[
\frac{\mu}{\partial s} \frac{\partial f}{\partial s} + \frac{1-\mu^2}{2\sigma} \frac{\sigma}{\partial \mu} \frac{\partial f}{\partial \mu} = \frac{A N_e}{E^2} \left[ E \frac{\partial \varphi}{\partial E} + \frac{\partial}{\partial \mu} \left( \frac{1-\mu^2}{2\mu} \frac{\partial \varphi}{\partial \mu} \right) \right] (1)
\]

where \( \mu \) is the cosine of the pitch angle, \( \varphi = E/\mu \) is the distribution function of the superthermal electrons, \( \phi \) is the energy of the electrons, \( A = 2\pi e^2 \ln \Lambda = 2.6 \times 10^{-12} \text{cm}^2 \text{eV}^2 \), \( \sigma(s) = B(s_0)/B(s) \), \( B(s_0) \) and \( B(s) \) are the magnetic field intensities at the boundary between the ionosphere and plasmasphere and a distance \( s \) along a given field line, respectively, (there is no clear boundary between the ionosphere and plasmasphere, so some appropriate value needs to be selected for a given calculation). The terms on the left-hand side of equation (1) describe the change in the distribution of the electrons as they move in an inhomogeneous magnetic field, while the terms on the right-hand side correspond to the change in energy and pitch angle due to collisions with thermal electrons and ions.

When there are no collisions present and the source of electrons is the ionosphere, the distribution function at any point along the field line is uniquely determined by the distribution at the foot of the field, at \( s_0 \), because of the conservation of the magnetic moment (first adiabatic invariant). The relationship between the cosine of the pitch angle \( \mu \) at a given point \( s \) along the field line and that at \( s_0 \), written as \( \mu_{so} \), is

\[
\mu_{so} = \sqrt{1-\sigma(s)[1-\mu^2]} \quad (2)
\]
Given that $\mu_0$ is real, the range of pitch angles at $s$ is

$$1 - \frac{1}{\sigma(s)} \leq \mu^2 \leq 1$$  \hspace{1cm} (3)

Inequality (3) characterizes a family of trajectories in the $(s, \mu)$ plane, corresponding to region I in Figure 1. This figure clearly shows how the pitch angle decreases as the electrons move from the ionosphere toward the magnetic equator and then increases as they move down into the conjugate ionosphere. Particles corresponding to this region can move freely from one hemisphere to the conjugate one and have been denoted as "precipitating," "free," or "fly-through" electrons.

Electrons trapped by the geomagnetic field are defined by the inequality

$$\mu^2 \leq 1 - \frac{1}{\sigma(s)}$$  \hspace{1cm} (4)

These particles, which correspond to region II in Figure 1, move in the $(s, \mu)$ space in closed circuits, along trajectories $L_\Pi$. The reflection points $s_{ref}$ are uniquely determined by the value of the pitch angle cosine $\mu_o$ at the equator ($s=0$), which is

$$\mu_o = \mu \left[ 1 - \frac{1}{\sigma(s)} \right] \frac{1}{1 - \mu^2}$$  \hspace{1cm} (5)

where $\sigma_o = \sigma(0) = B(0)/B(0)$. The reflection point is determined from the condition $\mu=0$:

$$\sigma(s_{ref}) = \sigma_o (1 - \mu_o^2)$$  \hspace{1cm} (6)

In the collisionless case the parameters of the superthermal electrons in the two zones are independent, because there is no interchange of particles between regions I and II. The electron population in the trapped region is the result of direct production of superthermal electrons in this zone. In the plasmasphere these sources are negligible, therefore it is possible to assume that $f_{II} = 0$. In this case the spatial variations of the thermal electron heating rate due to these superthermal electrons can be written as $[Khazanov, 1979]$

$$Q_s = Q_s(s = \pm s_o) \left[ 1 - \frac{1}{1 - \sigma(s)} \right]$$  \hspace{1cm} (7)

In the derivation of (7) it was assumed that the superthermal flux is isotropic at $s=\pm s_o$ and the integration over $\mu$ took into account inequality (4).

The above discussion neglected the effects of collisions in the plasmasphere; taking into account collisions produce significantly different results, even when the conventional mean free path of the superthermal electrons ($\lambda \sim E^2/AN_e$) is greater than the characteristic length of the field tube ($h_o \sim \sigma/\sigma^2/ds$). This can be demonstrated by looking at the second term on the right-hand side of equation (1), which accounts for the pitch angle diffusion:

$$\frac{AN_e}{2E^2} \frac{\partial \left[ 1 - \mu^2 \right]}{\partial \mu} = \frac{AN_e f \mu}{E^2} \Delta \mu$$  \hspace{1cm} (8)

where $\Delta \mu$ is the width of the "fly-through" zone at $s=0$ ($\sigma=\sigma_o$). The value of $\Delta \mu$ can be written as

$$\Delta \mu = 1 - \mu_{ob}$$  \hspace{1cm} (9)

where $\mu_{ob}$ corresponds to that pitch angle at the equator which results in reflection at $s=s_o$. Equation (9) can be evaluated by using equation (6) and recognizing the fact that $\mu_{ob}$ is close to unity, giving

$$\Delta \mu = \frac{1}{2\sigma_o}$$  \hspace{1cm} (10)

Substituting this back into the right-hand side of (8) shows that pitch angle diffusion causes changes in the distribution function over a length scale of $\lambda/2\sigma_o \sim h_o$. Given that $\sigma_o >> 1$ for mid- and high-latitude field tubes, pitch angle diffusion needs to be taken into account even when the conventional mean free path is long compared to the field tube dimension. This can be also seen by recognizing the fact that the pitch angle width of region I is very narrow near the equator and thus even small angle deflections will result in electron trapping.

Equation (1) can be rewritten in terms of $\mu_o$ and $s$ giving

$$\frac{\partial f}{\partial s} = \frac{AN_e}{2E^2} \frac{\sigma_o \mu}{\sigma_o \mu_o} \frac{\partial \mu}{\partial s} \left[ 1 - \mu^2 \right] \frac{\partial f}{\partial \mu} + \frac{AN_e f}{E} \frac{\partial \mu}{\partial \mu}$$  \hspace{1cm} (11)

where $\mu$ is a function of $\mu_o$ and $s$. By introducing these new variables the second term on the left-hand side of equation (1)
is eliminated. The region over which \( f(s, \mu_o) \) is defined in terms of \( s \) and \( \mu \) is shown in Figure 2. In the rest of this paper we solve equation (11) by considering the symmetry between the two hemispheres, \( f(s_0^+, -\mu_o) = f(s_0^-, \mu_o) \). The long mean free path \( \lambda \gg H_o \), and field tubes with \( \sigma_o \gg 1 \).

3. SOLUTION OF THE KINETIC EQUATION: INFINITE TRAPPED (CAPTURE) ZONE APPROXIMATION

Krinberg and Matafonov [1978] assumed that superthermal electrons which have scattered from region I (the "fly-through region") into region II (the trapped region) will not scatter back out again. These authors calculated the plasmaspheric transparency \( \Pi \) defined as

\[
\Pi(E) = \frac{\int_0^1 \mu f(E, \mu, \sigma_o) d\mu}{\int_0^1 \mu f(E, -\mu, \sigma_o) d\mu} = \frac{\mu_{ob}}{\mu_o} \tag{12}
\]

where \( \mu_{ob} \) is pitch angle cosine at the boundary between the trapped and precipitating electrons. In the above definition of the transparency only untrapped electrons were considered, thereby neglecting the interaction between the fly-through and trapped electron regions. We shall examine the solution to the kinetic equation (11), using an infinite capture zone approximation, with the aim of comparing this solution with a more general treatment of the problem to be described in subsequent sections and to assess the effect of the approximation introduced by the assumption of an infinite volume for region II.

It is possible to neglect the second term on the right-hand side of equation (11) for electrons which are in or near region I, because pitch angle diffusion is much more important than energy loss [Khazanov et al., 1981]. Therefore under these conditions it is relatively easy to show that equation (11) can be written as

\[
\frac{\partial f}{\partial z} = \frac{\partial}{\partial x} \left( x \frac{\partial f}{\partial x} \right) \tag{13}
\]

where

\[
x = 1 - \mu_0
\]

\[
z = \left( \frac{E_n^2}{E^2} \right) x_{ob}
\]

\[
E_n^2 = 2A \int \sigma(s')N_e(s') ds'
\]

It was noted before that \( \mu_0 \) is close to unity, and this was used in obtaining equation (13). An examination of equation (11) shows that pitch angle diffusion becomes important over a length scale of \( \lambda/\sigma \), requiring that \( \sigma \gg 1 \) for significant pitch angle change; this in turn means that \( \mu \ll 1 \) (see equation (3)). The values of \( \mu_{ob} \) and \( x_{ob} \), parameters of the boundary between the free and trapped zones, can be determined from (6) to be

\[
\mu_{ob} = \sqrt{\frac{1}{\sigma_o}} = \frac{1}{2\sigma_o} \tag{14}
\]

\[
x_{ob} = 1 - \mu_{ob} = \frac{1}{2\sigma_o} \tag{15}
\]

Equation (13) is solved under the boundary condition

\[
f(z=0, x) = f^*(x) \Theta(x_{ob} - x) \tag{16}
\]

where \( f^*(x) \) is the distribution function of superthermal electrons, injected into the plasmasphere from the ionospheric level \( -s_o \) and \( \Theta \) is the step function. The other boundary conditions used are:

\[
f(z=0) \text{ bounded} \tag{17}
\]

\[
f(z, x \to \infty) \to 0 \tag{18}
\]

Boundary condition (18) requires some explanation, because the domain of definition of \( x \) is between 0 and 1. The use of an upper limit of infinity can only be justified in the case of sharp gradients of the distribution function with respect to \( \mu_o \). In the case under consideration here the region close to \( x=1 \) may be regarded as being infinitely far from the interval in \( x \) between 0 and \( x_{ob} \). The assumption that the particles injected from ionospheric levels will concentrate in the \( \mu_o \) plane near their original value of \( \mu_o \), does justify this assumption, sometimes referred to as the "infinite capture zone."

The solution to (13) can be written as (see Appendix A)

\[
f(x, z) = \int_0^x f^*(\zeta) G(x, \zeta, z) d\zeta \tag{19}
\]

where the Green function \( G(x, \zeta, z) \) is given by (A9) and \( \zeta \) is a dummy variable. As an example, let us consider that electrons are injected at \( s=-s_o \) in a direction parallel to the magnetic field, therefore

\[
f^*(\zeta) = \delta(\zeta) \tag{20}
\]

Substitution of (A9) and (20) into (19) for the distribution function yields

\[
f(x, z) = \frac{1}{z} \exp \left( -\frac{x}{z} \right) \tag{21}
\]

Equation (21) indicates that a strong gradient in pitch angle is present at locations where

\[
z = \frac{E_n^2}{2\sigma_0 E^2} \ll 1 \tag{22}
\]

It is inequality (23) that, in fact, defines the parameters of the field tubes and the energy of superthermal electrons, for which the infinite capture zone approximation holds true. This inequality corresponds to conditions when the mean free path is much longer than the plasma scale height.

Next we determine the expression for plasmaspheric transparency, when \( f^*(\zeta) \) is a weak function of \( \zeta \). First, we define \( f(z) \) to be the mean value of \( f(x, z) \) in the interval between 0 and \( x_{ob} \) for this case of slowly varying \( f^* \):

\[
\bar{f}(z) = \frac{1}{x_{ob}} \int_0^{x_{ob}} f(x, z) dx \tag{23}
\]

We now substitute into (23) equation (19) for \( f(x, z) \):

\[
\bar{f}(z) = \frac{1}{x_{ob}} \int_0^{x_{ob}} f^*(\zeta) G(x, \zeta, z) d\zeta dx = \int_0^{x_{ob}} G(x, \zeta, z) d\zeta dx \tag{24}
\]
Next we use equation (12), which defines the transparency $\Pi$ and recognize that between $g_0$ and $1$, $g_0 - 1$; therefore we can write

$$\tilde{t}^+ \Pi = \int_{g_0}^{1} \mu f(E, \mu, \gamma) d\mu = \int_{g_0}^{1} f(E, \mu, \gamma) d\mu = \tilde{t}(z)$$

Recalling the relation

$$\int_{g_0}^{1} J_0(2k\sqrt{x}) dx = \frac{\sqrt{x_0}}{k} J_1(2k\sqrt{x_0})$$

the expression for the transparency can be written as

$$\Pi(z) = 2 \int_{g_0}^{1} \exp(-k^2 z) J_1(2k\sqrt{x_0}) dx = 1 - \Phi(\frac{3}{2}, \frac{4z}{x_0}; \frac{4x_0}{z}) \exp(-\frac{4x_0}{z})$$

where $\Phi$ is the degenerate hypergeometric function [Gradshteyn and Ryzhik, 1980]. In this case, where $\gamma^*$ was assumed to be a weak function of angle, the capture zone approximation is valid only when inequality (22) is satisfied; this is a limitation on calculating the transparency given by equation (27). The numerical values and specific variations described by equation (27) are discussed in section 5.

4. THE DISTRIBUTION FUNCTION OF SUPERTHERMAL ELECTRONS IN THE PLASMAPHERE

In this section the transport of superthermal electrons in the plasmasphere is studied, but this time the trapped/capture zone is considered to be finite, that is, scattering back from the trapped (II) into the fly-through (I) zone is taken into account. Let us denote the distribution function of the electrons in zone I as $f_1$ and obtain a solution for it from equation (13), using the initial condition given by equation (16). The boundary condition at $x=0$ ($l_{to}=(+/-)l$) is the boundedness of $f_1$. The boundary condition at $x=x_{ob}$ is assumed to be the function $f_0(E)$, which is taken to be constant along the trajectory $x=x_{ob}$ [Khazanov, 1979]. We can write $f_1$ as follows:

$$f_1(x,z) = \Psi(x,z) + f_0$$

Using equation (13), this leads to

$$\frac{\partial \Psi}{\partial z} = \frac{\partial}{\partial x}(x \frac{\partial \Psi}{\partial x})$$

The boundary conditions can be summarized as

$$\Psi(x, z=0) = \gamma^*(x) - f_0$$

$$\Psi(x, z=0) \text{ bounded}$$

$$\Psi(x, z=x_{ob}) = 0$$

Using separation of variables to solve equation (29) yields

$$f_1(x,z) = f_0 + \sum_{n=1}^{\infty} \exp(-\frac{\alpha_n^2 z}{4x_{ob}}) J_0(\alpha_n \sqrt{x_{ob}}) \int_{x_{ob}}^{\infty} f^+(x') J_0(\alpha_n \sqrt{x'}) dx'$$

where $J_0(x)$ and $J_1(x)$ are Bessel functions and the $\alpha_n$ values are the eigenvalues of the initial problem, defined by

$$J_0(\alpha_n) = 0; \quad \alpha_n = \frac{n}{4}(4n-1); \quad n=1,2,\ldots.$$ (34)

The pitch angle distribution of ionospheric electrons (generally photoelectrons or backscattered secondary electrons) escaping into the plasmasphere is generally characterized by a weak variation in pitch angle over the interval $x_{ob} < \alpha < 1$ [Khazanov and Gefen, 1982]. Given this characteristic of $\gamma^*$, it can be moved outside the integral in (33), and this simplifies the expression for $f_1$ to

$$f_1(x,z) = f_0 + \sum_{n=1}^{\infty} \exp(-\frac{\alpha_n^2 z}{4x_{ob}}) J_0(\alpha_n \sqrt{x_{ob}}) \frac{2(\tilde{t}^+ - f_0)}{\alpha_n J_1(\alpha_n)}$$

where

$$\tilde{t}^+ = \frac{1}{x_{ob}} \int_{x_{ob}}^{x_0} f^+(x') dx'$$

is the pitch angle averaged distribution function of the superthermal electrons injected into the plasmasphere from the ionosphere. A similar pitch angle averaging of equation (35) for $f_1$ gives

$$\tilde{f}_1(x) = f_0 + (\tilde{t}^+ - f_0) \sum_{n=1}^{\infty} \frac{4}{\alpha_n^2} \exp(-\frac{\alpha_n^2 x}{4x_{ob}})$$

where

$$\gamma_{s} = \frac{E^2}{E_n(s)}$$

Given that $\alpha_n^2 << \alpha_0^2 << \ldots$, the main contribution to the infinite sum in equation (37) is the first term when $z/x_{ob}$ is large. When $z/x_{ob}$ is small, the contribution from the rest of the terms is appreciable, because

$$\frac{4}{\alpha_1^2} = 0.72; \quad \sum_{n=1}^{\infty} \frac{4}{\alpha_n^2} = 1$$

Therefore in order to get an approximation for arbitrary $z/x_{ob}$ it is assumed that $4/\alpha_1^2 \approx 1$. Thus (37) becomes

$$\tilde{f}_1(y_s) = f_0 + (\tilde{t}^+ - f_0) \exp(-\frac{1}{y_s})$$

where

$$y_s = \frac{E^2}{E_n(s)}$$

Next we look at the solution to equation (11) in region II, the so-called trapped or capture zone. If we assume that $f(s, l_{to})$ varies only slightly along its trajectory (denoted by $L_H$ in Figure 2) [Khazanov, 1979], integrating equation (11) over this trajectory, makes the left-hand side of the equation (39) approach zero, because the distribution function is continuous. Therefore the averaged kinetic equation for this capture zone can be written as [Polyakov et al., 1979; Khazanov et al., 1979a]

$$\frac{\partial}{\partial l_{to}} \left[ a(\mu_o) [1 - \mu_o^2] \frac{\partial f}{\partial l_{to}} \right] + 2B(\mu_o) E \frac{\partial f}{\partial E} = 0$$

where

$$a(\mu_o) = \int_{0}^{\infty} \frac{\sigma_o^2 \sigma_o}{\alpha_o} \mu N_e(s) ds.$$
and $f_{\Pi}$ is the mean value of $f(s, \mu_o)$ along the trajectory $L_{\Pi}$. If the ratio $s(\mu_o)/B(\mu_o)$ is assumed to be independent of $\mu_o$ and its value is taken to be 0.5, equation (39) can be simplified to

$$\frac{\partial}{\partial x} (x \frac{\partial f_{\Pi}}{\partial x}) + 2E \frac{\partial f_{\Pi}}{\partial E} = 0 \quad (40)$$

It has been shown both analytically and numerically that the solutions to equations (39) and (40) are very similar in the region near the boundary between the trapped and fly-through zone [Khazanov, 1979].

Next we will obtain the distribution function, $f_{\Pi}$, in the trapped zone, using equation (40). This equation is not appropriate in the region $x \rightarrow 1$ ($o \rightarrow 0$), but most of the trapped electrons are found in the region near $x=x_{ab} \ll 1$ ($\mu_o=\mu_{ab}$), therefore this limitation will not cause a problem. The boundary conditions to be used to solve equation (40) are

$$f_{\Pi}(x=x_{ab}) = f_o, \quad \frac{\partial f_{\Pi}}{\partial x} \bigg|_{x=x_{ab}} = 0, \quad f_{\Pi}(E \rightarrow \infty) \rightarrow 0 \quad (41)$$

We write $f_{\Pi}(x,E) = \Psi(x,E) + f_o$, and the new boundary problem takes the form

$$\frac{\partial}{\partial x} (x \frac{\partial \Psi}{\partial x}) + \frac{E}{2} \frac{\partial \Psi}{\partial E} + \frac{E}{2} \frac{\partial f_o}{\partial E} = 0$$

$$\Psi(x=x_{ab}) = 0, \quad \frac{\partial \Psi}{\partial x} \bigg|_{x=x_{ab}} = 0, \quad \Psi(E \rightarrow \infty) \rightarrow 0 \quad (42)$$

The general solution to equation (42) is obtained in Appendix B, where it is also shown that a good approximation to the solution is

$$f_{\Pi}(E, x) = f_o \ln x + d \left(1 - \frac{\ln x}{\ln x_{ab}}\right) y^d \int_{y'}^{\infty} f_o(y') \frac{dy'}{y'/\gamma\left(1 + d\right)} \quad (43)$$

where $d=|\ln(1/x_{ab})|^{-1}$ and $y=E^2/E_{no}^{-2}$, where $y$ is the same variable as introduced with equation (38), with $E_{no}=E_{no}(s_0)$. The solutions we obtained for the distribution functions in both the trapped (capture) and fly-through (free) regions, equations (43) and (38), respectively, contain the arbitrary function $f_o(E)$. Therefore we will now see how an expression for $f_o$ can be obtained.

A continuity equation for the superthermal electrons is obtained by integrating the kinetic equation (1) with respect to $\mu$ from -1 to 1, which is

$$\frac{\partial n_{Eo}}{\partial s} = \frac{A N_s \partial \eta_{Eo}}{E} \frac{\partial E}{\partial E} \quad (44)$$

where

$$\eta_{Eo} = \int_{-1}^{1} \mu f(s) \, d\mu \quad n_{Eo} = \int_{-1}^{1} f(s) \, d\mu$$

Integrating (44) with respect to $s$, from $s_0$ to $+s_0$, and taking into account the fact that $\sigma_o >> 1$ and that the main contribution to the integral on the right-hand side comes from the value of $n_{Eo}(s)$ at the top of the flux tube, $n_{Eo}$ [Khazanov and Gefan, 1982], we get

$$\eta_{Eo}^+ = \frac{E}{\gamma} \int_{y'}^{\infty} f_o(y') \frac{dy'}{y'/\gamma\left(1 + d\right)} \quad (45)$$

Inequation (45) the rate at which superthermal electrons enter the plasmasphere from the ionosphere, $n_{Eo}^+$, is the mean of that entering and escaping. Note that in the determination of the total superthermal electron density at the magnetic equator, $s=0$, it was assumed that the contribution from those in the fly-through region is negligible. Using these various approximations in equation (43), the following expression can be obtained for $n_{Eo}$:

$$n_{Eo} = 2 \int_{s=0}^{s=1} f(s) \, d\mu = 2 \int_{s=0}^{s=1} f_{\Pi}(s) \, ds \quad (46)$$

Combing equations (38), (45), and (46), we can obtain the following differential equation involving $f_o$:

$$\frac{\partial \Phi_o}{\partial y} - \varphi \phi_o = 2 \frac{\partial \tilde{f}^+}{\partial y} \quad (47)$$

where

$$\Phi_o = (\tilde{f} - f_o)(y \xi^2 + 2d), \quad \xi(y) = \frac{(1 - e^{-1/y})}{2}, \quad \varphi = \frac{d\xi}{y\xi^2 + 2d}$$

Given that $d << 1$ and that the essential variations of $\xi(y)$ take place for $y>1$, $\phi(y)$ can be written as $d/(4d+y)$. The expression for $f_o$ can then be written as

$$f_o = \tilde{f}^+ - \frac{y + 4d}{y + 4d + 2d} \int_{y}^{\infty} \tilde{f}^+ \frac{dy'}{y'^y + 4d + dy'^y} \quad (48)$$

Equations (38), (43), and (48) are the necessary relations for determining the distribution functions of the superthermal electrons in the plasmasphere. However, one should note that in order to obtain the necessary solutions one needs to know the distribution function of the electrons escaping from the ionosphere, but the self-consistent calculation of that requires the plasmaspheric distribution at the interface. Therefore a self-consistent simultaneous solution for both the ionosphere and plasmasphere is needed. This can be achieved either by iteratively matching analytic or numerical ionospheric solutions with the analytic solutions described above or by self-consistent analytic or numerical solutions for both the ionosphere and plasmasphere. The latter is possible but will not be presented in this paper; it will be the subject of a later paper. We will now examine the nature of the plasmaspheric electrons using an assumed form for the ionospheric electrons entering the plasmasphere.

5. Plasmaspheric Transparency and Heating Rate

Let us consider the example in which the electrons moving into the plasmasphere from the ionosphere have an exponential energy distribution $[1 - \exp(-E/E_{no})]$. In this case, equation (48) becomes

$$f_o = \frac{y + 4d}{y + 4d} \tilde{f}^+ \quad (49)$$

where $\Delta=1$ when $E_o>>E_{no}$ and $D=\exp^2/\exp^2_{no}$ when $E_o<<E_{no}$

We can now write an expression for the plasmaspheric transparency $\Pi$ which we defined earlier (equation (12)) as the
ratio of superthermal electrons escaping from the plasmasphere over those entering. In the present approximation the transparency becomes

$$\Pi = \frac{\bar{\tau}^+}{\bar{\tau}^-} = \frac{2\delta(y+4d\Delta)}{y+4d} + \exp(-1/y)$$  (50)

Equation (50) clearly shows that the transparency approaches unity as y gets large, or, in other words, high-energy electrons pass through the plasmasphere, without significant absorption. The transparency also remains finite at low energies. The solid line in Figure 3 shows the value of the transparency for d=0.18, γ=5, and Δ=2.5x10^-3, which can be compared to the dashed line, which corresponds to values obtained from (50) with the Δ term neglected, and the dashed-dotted line which corresponds to the infinite trapped (capture) zone approximation. It can be clearly seen that this infinite trapped zone approximation has only a limited range of applicability, and is inaccurate at low energies. Figure 4 shows the calculated transparencies for a number of different geomagnetic latitudes obtained from both the analytic expression given by equation (50) and the numerical solution of equation (1) [Khazanov et al., 1979b]. In the calculations the thermal electrons and ions were assumed to be in diffusive equilibrium along the field line with N_e(x_o)=5x10^3 cm^-3 and T_e+T_i=7x10^3 K. Table 1 gives the various parameters which were used in evaluating the analytic solution given by equation (50). Figure 4 clearly indicates that the analytic approximation agrees well with the numerical solution of equation (1) for the same conditions.

Next we calculate the heating rate resulting from the passage of these superthermal electrons through the plasmasphere [Khazanov, 1979].

$$Q_e = 2\pi A N_e \int_0^\infty f(E,\mu) d\mu dE$$  (51)

Using equations (38) and (49), the distribution function for the free electrons can be written as

$$\bar{f}_t(y) = \frac{y+4d\Delta}{y+4d} \left[ \frac{1}{1-\exp(-1/y)} + \exp(-1/y) \right]$$  (52)

Combining equations (43) and (49), we get the distribution function of the trapped electrons:

$$Q_{e\text{f}} = \int \frac{dE_2}{E_2} Q_{e\text{f}}$$  (53)

where $W_{a,b}$ is the Whittaker function [cf. Gradshteyn and Ryzhik, 1980]. It can be shown using equation (53) that most of the superthermal electrons in the trapped zone, having energies $E>>E_o$, concentrate near the boundary trajectory, $x=x_{ob}(\mu_{ob}=\mu_{ob})$ (see Figure 5). Using equations (52) and (53), we can calculate the separate heating rates due to both the free and trapped electrons, $Q_{ef}$ and $Q_{e\text{f}}$, respectively, which are

$$Q_{ef} = \frac{1}{E_2} \int \frac{dE_2}{E_2} Q_{ef}$$  (54)

$$Q_{e\text{f}} = \frac{1}{E_2} \int \frac{dE_2}{E_2} Q_{e\text{f}}$$  (55)

where $Q_{e\text{f}}$ is the heating rate at the base of the plasmasphere, $\eta_E := 2(E_2/E_o)$, $E_2 = E_{no}$, and $E_2 = E_{no} - E_n^2$. Adding equations (54) and (55) together gives the expression for the total heating rate:

$$Q_e = Q_{ef} + Q_{e\text{f}} = Q_{ef}$$

where $Q_{ef}$ is the heating rate at the base of the plasmasphere, $\eta_E := 2(E_2/E_o)$, $E_2 = E_{no}$, and $E_2 = E_{no} - E_n^2$. Adding equations (54) and (55) together gives the expression for the total heating rate:

$$Q_e = Q_{ef} + Q_{e\text{f}} = Q_{ef}$$

Table 1. Parameters Used for Evaluating Equation (50)

<table>
<thead>
<tr>
<th>φ</th>
<th>$R_o \cdot 10^4$, km</th>
<th>$\sigma_0$</th>
<th>2d</th>
<th>$E_{no}$, eV</th>
<th>Δ</th>
<th>$E_n^2$, eV</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>1.3</td>
<td>12.6</td>
<td>0.79</td>
<td>15</td>
<td>0.30</td>
<td>10</td>
</tr>
<tr>
<td>55</td>
<td>2.1</td>
<td>48.7</td>
<td>0.44</td>
<td>30</td>
<td>0.10</td>
<td>10</td>
</tr>
<tr>
<td>60</td>
<td>2.7</td>
<td>115.0</td>
<td>0.36</td>
<td>59</td>
<td>0.03</td>
<td>10</td>
</tr>
</tbody>
</table>
Fig. 5. Plot of the ratio of the distribution function at $x=1$ over that at $x=x_{ob}$ in the trapped zone, as a function of the energy. The solid lines correspond to the approximate analytic result given by equation (B7) and the dashed line to the "exact" numerical solution obtained by Khazanov et al. [1979b].

If we take $\sigma_{\theta}>>1$ (this is easily satisfied for mid- and high-latitude field tubes), we obtain for the variation of the thermal electron heating, due to the passage of superthermal electrons along a geomagnetic field line, the following simple expression:

$$Q_c = Q^{-} + \frac{E^2}{E_n^2} \left[ \frac{1}{\sigma} - \frac{1}{\sigma_n} \right] + \frac{2}{\sigma^2} \exp(-\eta^+).$$

The above result was obtained for the flux entering the plasmasphere, $f^+$, having an exponential form ($\exp(-E/E_n)$). It can be shown that if $f^+$ is given by a delta function ($\delta(E-E_n)$) and $d<<1$, equations (53) and (57) also hold in the limit of $d<<1$. This means that if the exponential approximation for $f^+$ is not a good one, the input flux can be approximated by a series of delta functions and the corresponding distribution function for the trapped population and heating rate can be "built up" using equations (53) and (57). The approximation of $d<<1$ corresponds to a very wide trapping zone, which is satisfied in general.

Finally, we can obtain expressions for the integrated heating rates along the field line due to the trapped and fly-through electrons:

$$Q_n = \int ds \frac{Q_n}{N_n}$$

Integrating equation (55), we obtain

$$Q_{s\theta T} = Q^{-} \cdot \frac{E^2}{2A \sigma_n}$$

where

$$N_n = \int_{x_{ob}}^{x_{ob}} \frac{N_n}{\sigma n} dx$$

In order to calculate $Q_{s\theta T}$ the function

$$\sigma \left[ 1 - \sqrt{\frac{1}{\sigma_n}} \right]$$

can be assumed to be equal to 0.5 and the function $\eta^+(s)$ written as $F_{\eta^+}$, where

$$F = \frac{16 A N_n \sigma_n^{-1/3}}{E_n^2}$$

Under these assumptions the integrated heating rate due to superthermal electrons in region I is

$$Q_{s\theta T} = Q^{-} \cdot \frac{3E^2}{16A \sigma_n AN_n}$$

Comparison of equations (59) and (60) indicates that the main contribution to the integrated heating rate comes from the trapped superthermal electrons.

6. CONCLUSIONS

We derived analytic expressions for the distribution functions of superthermal electrons in the plasmasphere. These results were then used to obtain simple expressions for the transparency of the plasmasphere to the passage of these electrons and the resulting heating of the thermal electron population. As expected the transparency decreases with decreasing energies and the thermal plasma heating is due mainly to the trapped superthermal population. These results will be very useful in future ionospheric model calculations, as it will allow the inclusion, in a quantitative manner, of the energy deposition in the plasmasphere and subsequent heat input into the ionosphere. Also, the expression for transparency, obtained in this work, will make possible quantitative calculations of the effects of conjugate photoelectrons. Clearly, a more accurate and self-consistent set of calculations would require the numerical and simultaneous solutions of the electron transport equations in both the ionosphere and plasmasphere. Such calculations will be carried out in the future. However, the use of the results presented here will, by themselves, lead to better and more quantitative model calculations.

APPENDIX A

The method of separation of variables is used to obtain a solution to equation (13); thus we write

$$f(x,z) = X(x)Z(z)$$

This in turn leads to

$$X(x) = C_{1k} J_0(2k\sqrt{x}) + C_{2k} N_0(2k\sqrt{x})$$

$$Z(z) = \exp(-k^2z)$$

where $k$ is the separation constant and $C_{1k}$ and $C_{2k}$ are constants. $J_0$ and $N_0$ are the zero-order Bessel and Neuman functions. The second boundary condition, equation (17), requires that $C_{2k}$ be zero, because $N_0(x)$ goes to infinity as $x \to \infty$. Thus the solution to equation (13) can be written as

$$f(x,z) = \int_{0}^{\sqrt{x}} C(k) \exp(-k^2z) J_0(2k\sqrt{x}) dk$$

where $C(k) = C_{1k}$. Next, using the boundary condition given by equation (16), we get
Recalling the following relation,
\[ \int_0^\infty J_o(2k\sqrt{x})J_o(2\sqrt{x})dx = \frac{1}{2k} \delta(k-p) \]  
(A6)
we can obtain \( C(k) \) by first multiplying (A5) by \( J_o(2p+x) \) and then integrating it over \( x \). This leads to
\[ C(k) = 2k \int_0^\infty f^+(\xi)J_o(2k\sqrt{\xi})d\xi \]  
(A7)
Substituting (A7) into (A4) and introducing the Green function yields
\[ G(x, \zeta, z) = 2 \int_0^\infty k \exp(-k^2z)J_o(2k)J_o(2k)dk \]  
(A8)
Integrating the above given Green function yields
\[ G(x, \zeta, z) = \frac{1}{z} \int_0^\infty \left( \frac{2\sqrt{\xi}}{z} \right) \exp\left[-\frac{x+\xi}{z}\right] dx \]  
(A9)
where \( I_0 \) is the modified Bessel function.

APPENDIX B

The boundary conditions in (41) are equivalent to a statement on the symmetry condition on the pitch angle distribution of superthermal electrons in the trapped region \( f^+(\mu_o) = f^-(\mu_o) \), because the initial equation (39) varies as the sign \( m_o \) does. The eigenfunctions \( M_n(x) \) of the solution to equation (42) are given in terms of Bessel and Neuman functions as
\[ M_n(x) = N_1(2Kn)J_o(2Kn\sqrt{x}) - J_1(2Kn)N_0(2Kn\sqrt{x}) \]  
(B1)
where the \( K_n \) values are the eigennumbers (n=0,1,2,...) determined from the equation \( M_n(x_{ob}) = 0 \). The system of functions given by (B1) are orthogonal in the interval \( x_{ob} < x < 1 \). The square of the norm is
\[ A_n = \int_0^{x_{ob}} \left[ M_n(x) \right]^2 dx = x_{ob} B_n M_n^2 \]
where
\[ B_n = \frac{1}{\sqrt{x_{ob}}} M_n^0 - M_n^1 \]
\[ M_n^0 = J_1(2Kn)N_0(2Kn\sqrt{x_{ob}}) - N_1(2Kn)J_0(2Kn\sqrt{x_{ob}}) \]  
(B2)
We can write \( \Psi(x,E) \) as an infinite series using \( M_n(x) \):
\[ \Psi(x,E) = \sum_{n=0}^{\infty} C_n(E)M_n(x) \]  
(B3)
The expansion coefficients \( C_n \) are obtained by substituting (B3) into (42), which gives
\[ \frac{\partial C_n}{\partial E} - 2K^2 C_n = - \frac{1}{A_n} \frac{\partial}{\partial E} \int_0^{x_{ob}} M_n(x)dx \]  
(B4)
Using these results, we can write the following expression for the distribution function of superthermal electrons in the trapped zone as
\[ f^+_n = f_n + \frac{1}{\sqrt{x_{ob}}} \frac{\partial f_n}{\partial E} \sum_{n=0}^{\infty} \left( \frac{E}{E_0} \right)^{2K^2} M_n(x) \frac{dE}{dE_0} \]  
(B5)
Equation (B5) can be simplified if \( x_{ob} < 1 \). Under these conditions the \( K_n \) and \( C_n \) coefficients can be written as
\[ \frac{1}{K^2} = \ln \left( \frac{1}{\gamma^2 K^2 x_{ob}} \right) = \ln \left( \frac{1}{x_{ob}} \right) \quad C_n(x) = \frac{\pi}{2} \left[ \ln \left( \frac{\gamma^2 K^2 x_{ob}}{n+1} \right) \right] \]  
(B6)
where \( \gamma \) is the Euler constant. The inequalities given below follow from (B6):
\[ K_o << K_n \quad \Rightarrow \quad C_n(x) >> C_{n+1} \]  
Therefor in evaluating (B5) only the first term in the series has to be considered, thus giving
\[ f^+_n(E,x) = f_n \frac{\ln x}{\ln x_{ob}} + \int_0^{\infty} \left( 1 - \frac{\ln x}{\ln x_{ob}} \right) y^2 dy \]  
(B7)
where \( d = K_o^2 = (1/x_{ob})^{-1} \) and \( y \) is a dummy variable. Here we also introduced the variable \( y = E^2/E_{no}^2 \), which is consistent with \( y_* \), introduced in connection with equation (38); \( E_{no} = E_{no}(\kappa_0) \).

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