THE INAPPLICABILITY OF SPATIAL DIFFUSION MODELS FOR SOLAR COSMIC RAYS

A. J. OWE N S
Bartol Research Foundation of The Franklin Institute, University of Delaware

AND

T. I. GOMBOSI
Central Research Institute for Physics, Budapest, Hungary

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ABSTRACT

Because of the rapid temporal evolution of the particle density, the interplanetary propagation of solar cosmic rays cannot be accurately described by a spatial diffusion equation. With the use of a simple numerical integration scheme, we show that the pitch-angle scattering equation gives intensity-time profiles and anisotropies very different from the spatial diffusion results in many cases. The reasons for the failure of the diffusion limit, and an explanation for why some temporal profiles of solar particle events look diffusive, are given.

subject headings: cosmic rays: general — Sun: flares

I. INTRODUCTION

One of the outstanding unsolved problems in cosmic-ray physics is the large discrepancy between the scattering mean free paths predicted by diffusion theory and those observed at nonrelativistic energies in impulsive solar flare events (Jokipii 1979; Van Hollebeke 1979). Since the "observed" mean free paths are usually derived using a spatial diffusion equation (e.g., Zwickl and Webber 1977a, b), while the "theory" is typically quasi-linear theory (QLT) based on the slab model of interplanetary magnetic field fluctuations (Jokipii 1971), there are several important intermediate steps in the logical argument that could in principle be the cause of the quoted discrepancy.

At the most fundamental level, Klimas and Sandri (1971, 1973) questioned whether magnetic-field scattering produces the pitch-angle scattering equation given by QLT or the Fokker-Planck formalism (Jokipii 1966, Hasselmann and Wilberenz 1968). The deficiencies of QLT, such as the lack of scattering at 90° pitch angle, are mostly corrected when higher order terms in the quasi-linear expansion are included (Völk 1973; Goldstein 1976), so this is probably not the cause of the discrepancy.

A second possibility is that the form of the pitch-angle scattering equation (as our eq. [1] below) is correct, but that the nonlinear terms give a value for the pitch-angle scattering coefficient $D_\theta$ that differs significantly from QLT. Recent Monte Carlo simulations (Jones, Kaiser, and Birmingham 1973; Kaiser, Birmingham, and Jones 1978; Owens and Gombosi 1980) show that this is not the case. In the slab model of turbulence, the actual pitch-angle scattering coefficient is accurately predicted by QLT (for pitch angles away from 90°) even with large magnetic field fluctuations ($\delta B/B \sim 0.7$).

Another possibility is that the slab model is an inappropriate representation of the interplanetary magnetic field (IMF) fluctuations. Morfill, Lee, and Völk (1976) emphasized the role of medium-scale variations of the IMF, on scales large compared with the resonant wavelength but small compared to 1 AU. Recently Goldstein (1980) suggested that the IMF fluctuations appear more like a randomly rotating, fixed-length vector than two independently varying transverse fields. The influence of such a configuration on $D_\theta$ has not yet been tested by simulations.

Given the pitch-angle scattering equation, one can obtain a spatial diffusion equation under quasi-static and quasi-isotropic conditions. This will be discussed in detail in the next section. Our suggestion will be that it is here that the chain of reasoning breaks down, and that a spatial diffusion equation derived via the "diffusion limit" is inappropriate for the impulsive injection characteristic of solar flare events.

A final possible cause of the discrepancy between observed and calculated mean free paths is that additional effects are important: coronal storage and propagation, adiabatic deceleration and convection, particle drifts, and adiabatic focusing. Because of the short lifetime of solar flare events, drifts are unlikely to be important. Convection and adiabatic deceleration can be included into the diffusion models (Scholer 1976; Owens 1979), as can coronal transport (Reinhart and Wilberenz 1974); and the recent analyses (Zwickl and Webber 1977a, b; Hamilton 1977) do include these effects. Adiabatic focusing (Roelof 1969) has been included in a propagation model by Earl and his co-workers (Earl 1976; Ma Sung and Earl 1978; Bieber 1977).

Using the numerical integration scheme of Ng and Wong (1979), we have recently given results for the focused pitch-angle transport of solar cosmic rays (Gombosi and Owens 1980). Our conclusion was that fits of data to a simple spatial diffusion model give an "ob-
served" scattering mean free path considerably larger than that actually experienced by nonrelativistic particles. Increasing the focusing length to an arbitrarily large value led us to conclude that the discrepancy was not due to focusing. We show below that the discrepancy is of a much more fundamental nature: the diffusion limit is inapplicable to solar flare events, because of the impulsive nature of the phenomenon. The intensity-time profiles, while "looking" like solutions to the spatial diffusion equation, give "fitted" scattering mean free paths that differ only by about a factor of 2 over the entire range from nearly scatter-free conditions to those in which many tens of mean free paths are traversed by the particles between the source and the observer.

II. THE DIFFUSION LIMIT

The scattering of particles by magnetic inhomogeneities yields a pitch-angle scattering equation (Jokipii 1966; Hasselmann and Wibberenz 1968)

$$\frac{\partial n}{\partial t} + w \mu \frac{\partial n}{\partial z} = \frac{\partial}{\partial \mu} \left( D_\mu \frac{\partial n}{\partial \mu} \right)$$

(1)

for the cosmic-ray density $n(\mu, z, t)$ in phase space. Here $z$ is the spatial coordinate whose direction points along the average magnetic field, $w$ is the particle speed, $\mu = w_z/w$, and $D_\mu$ is the pitch-angle scattering coefficient. For quasi-static and quasi-isotropic conditions, the density can be expressed as

$$n(\mu, z, t) = U(z, t) + n_1(\mu, z, t),$$

(2)

where $U$ is the omnidirectional intensity and $n_1$ is the anisotropic part of the distribution function. Assuming $n_1$ is odd in $\mu$, taking separately the terms that are even and odd in $\mu$, one has

$$\frac{\partial U}{\partial t} + w \mu U = \frac{\partial}{\partial \mu} \left( D_\mu \frac{\partial n_1}{\partial \mu} \right),$$

(3)

and

$$\frac{\partial n_1}{\partial t} + w \mu \frac{\partial n_1}{\partial z} = \frac{\partial}{\partial \mu} \left( D_\mu \frac{\partial n_1}{\partial \mu} \right).$$

(4)

The diffusion limit is then obtained by neglecting the time-derivative $\partial n_1/\partial t$ in equation (4) with respect to the others, and one then obtains a spatial diffusion equation (Jokipii 1966; Earl 1973; Luhmann 1976)

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial \mu} \left( K \frac{\partial U}{\partial \mu} \right),$$

(5)

where the spatial diffusion coefficient $K$ is given by

$$K = \left( \frac{w^2}{4} \right) \int_0^1 (1 - \mu^2)^2 \frac{D_\mu}{\partial \mu} d\mu.$$  

(6)

The corresponding scattering mean free path is defined as

$$\lambda = 3K/w.$$

In the solar system, with the magnetic irregularities blown past a stationary observer by the solar wind with speed $V$, the effects of adiabatic deceleration and convection must also be included. The adiabatic focusing of particle trajectories by the diverging interplanetary magnetic field lines must also be taken into account. These effects can be added by additional terms in equations (1)-(6) (e.g., Jokipii 1971; Luhmann 1976), but they introduce more parameters and make an analysis of the individual effects of each process difficult to disentangle. For our discussion here, we will neglect these effects and will concentrate on the reduction of the pitch-angle scattering equation to the spatial diffusion equation. Similarly, we will ignore drifts and cross-field diffusion. We will show that the steps leading from equation (1) to equation (5) are invalid in most cases for which there is an impulsive release of particles somewhere in the system.

It has long been recognized that equation (5) must be inapplicable for "scatter-free" conditions in which the mean free path $\lambda$ is large compared with the macroscopic scales of interest in the problem (e.g., Earl 1973). For little scattering, the $D_\mu$ term in equation (1) can be neglected, and then the solution of equation (1) is readily obtained by characteristics:

$$n(\mu, z, t) = F(z - w_0 t),$$

(7)

where $F(x)$ is any function of $x$. Equation (7) simply expresses the rectilinear transport of particles with velocity $w_0 = w$. Since obviously particles cannot "diffuse" along $z$ more rapidly than their velocity $w$, this clearly means that the diffusion equation (5) will be inaccurate for cases in which it predicts evolution with a "speed" more rapid than $w$. For impulsive release of particles from the origin into infinite space at $t = 0$, if $K$ is constant, the solution to (5) is simply

$$U(z, t) = (\text{const.})^{-1/2} \exp(-z^2/4Kt).$$

(8)

The spatial extent of the particles moves approximately as $z = (4Kt)^{1/2}$, so the "speed" of diffusion is approximately $(K/t)^{1/2}$. For this speed to be less than the particle speed $w$, one must consider only times

$$t \leq t^* \equiv \lambda/w.$$

Clearly, if equation (8) predicts that many particles arrive at $z$ before the time $t^*$, then the spatial diffusion equation is inapplicable.

Although it is recognized that the diffusive idealization is inapplicable if there is too little scattering, as discussed above, we will demonstrate below that it is also inappropriate for impulsive injection if there is too much scattering as well. The qualitative picture is quite simple. Consider impulsive injection at a location $z = L/20$, with diffusion in the region bounded by $0 \leq z \leq L$. If the scattering is very rapid, isotropization will take place quite rapidly while spreading in $z$ will take place more slowly. One can then develop a nearly isotropic distribution of particles which are not yet in equilibrium because large spatial gradients exist. This scenario does not contradict Liouville's theorem (that the steady-state distribution is isotropic) because in a model with impulsive injection and an absorbing boundary anywhere the steady state is reached only when all particles have left the volume! For solar flare propagation, the steady state is never reached, as particles flow past Earth and eventually escape into the interstellar medium. Even though the pitch-angle distribution may become quasi-isotropic, there are still large spatial gradients. In equation (4), for example, this could result in the right-hand side being
negligibly small (since $\partial n_1/\partial \mu$ approaches 0) even though $D_\mu$ is large. Then neglecting the right-hand side of (4), taking the partial time derivative of (3), and eliminating $n_1$, we get the wave equation
\begin{equation}
\partial^2 U/\partial t^2 - (\mu \nu)^2 \partial^2 U/\partial z^2 = 0 .
\end{equation}

Equation (10) shows that there are conceivable circumstances for which the pitch-angle scattering equation (1) can give a wavelike rather than a diffusion-like transport of impulsively released particles. The solutions to the wave equation are arbitrary functions propagating with speed $w_z = w_\mu$ in the $z$-direction, just like equation (7). Although this rectilinear transport is characteristic of only a portion of the event, as will be shown by direct numerical integration below, it indicates that severe nondiffusive behavior can occur for impulsive injection even in the case of strong scattering.

III. NUMERICAL SOLUTIONS OF A SIMPLE MODEL

To demonstrate quantitatively the kinds of nondiffusive transport discussed in the previous section, we consider the following simple model. Particles are injected impulsively at time $t = 0$ and uniformly into the forward-going velocity cone ($\mu \geq 0$), at the location $z = L/20$. They propagate via the pitch-angle scattering equation (1) in the region bounded by $0 \leq z \leq L$, with absorbing boundaries at both ends. We choose for the pitch-angle scattering coefficient
\begin{equation}
D_\mu = D_0 (1 - \mu^2) ,
\end{equation}
which would be the functional form under QLT if the magnetic-field power spectrum were proportional to $k^{-2}$. This is the form for an isotropic distribution of magnetic scatterers. Next we define the dimensionless time $\tau = w t/L$ and distance $x = z/L$, so that equation (1) becomes
\begin{equation}
\partial n/\partial \tau = a \partial/\partial \mu [(1 - \mu^2) \partial n/\partial \mu] - \mu \partial n/\partial x .
\end{equation}
The only parameter in the model is $a = (L/w) D_0$.

From equation (6), one can calculate
\begin{equation}
K = w^2/(6 D_0) = wL/(6x) ,
\end{equation}
and the scattering mean-free path is $\lambda = L/(2x)$, so in terms of $x$,
\begin{equation}
\lambda_x = 1/(2x) .
\end{equation}

We solve equation (12) numerically, given the initial condition
\begin{equation}
n(x, \mu, \tau = 0) = (\text{const.}) H(\mu) \delta(x - x_0) ,
\end{equation}
where $H(x)$ and $\delta(x)$ are the Heaviside step function and Dirac delta function, respectively. The boundary conditions are free escape at the two spatial ends,
\begin{equation}
n(x, \mu = 0, \tau) = n(\mu, x = 1, \tau) = 0 ,
\end{equation}
and no flux "out" of the allowed pitch angle range $-1 \leq \mu \leq 1$.

\begin{equation}
D_\mu \partial n/\partial \mu = 0 \quad \text{at} \quad \mu = \pm 1 .
\end{equation}

We use an explicit finite-difference numerical scheme (Ng and Wong 1979) to solve equation (12) on a $21 \times 21$ grid ($x$ and $\mu$). The $\tau$ derivative is evaluated as a forward difference, and the $\mu$ derivatives are calculated from centered differences. (See, e.g., Gerald 1978 for a thorough discussion of numerical integration techniques.) For numerical stability in the $\mu \partial n/\partial x$ convective-like term, we use a forward difference in $x$ for $\mu > 0$ and a backward difference for $\mu < 0$. The boundary condition (15b) is not directly implemented as a "boundary condition" in the usual sense for numerical integration. Rather, the form of the pitch-angle scattering equation (with $D_\mu \to 0$ at $\mu = \pm 1$) automatically builds in the condition (15b) at the singular points $\mu = -1$ and $\mu = +1$. Our code uses forward and backward differences to calculate $\partial n/\partial \mu$ at $\mu = +1$ and $\mu = -1$, respectively. The point $\mu = 0$ is not singular and is treated as all interior points. We calculate the omnidirectional intensity $U(x, t)$ at a given spatial point and time by summing the distribution over all $21 \mu$ nodes.

Our numerical analysis routine has been subjected to an extensive set of tests to ensure its accuracy. The pitch-angle and spatial terms on the right-hand side of equation (12) have been individually suppressed, and the resulting numerical profiles are in agreement with the analytic solutions of these simpler problems. In the full equation (12), with instantaneous injection at $x = 1/2$, the total number of particles remains constant until the first particles reach the spatial "sinks" at $x = 0$ and $x = 1$. The algorithm has been coded in two languages and run on four different computers, in regular and extended precision, with no significant differences. Applying an explicit boundary condition based on evaluating the $\mu$ flux near the $\mu = \pm 1$ boundaries and requiring that the flux extrapolate to zero at the boundaries gives very similar results, indicating insensitivity to the boundary conditions. With the spatial boundaries at $x = 0$ and $x = 1$ held at two different densities, a steady state develops in which the anisotropy and spatial gradient are given by the usual steady-state relation $\delta = -(3K/\nu) x (U^{-1} \partial U/\partial x)$, with the diffusion coefficient given by equation (13b) (I. R. Jokipii 1980, private communication). We are thus confident that our numerical integration scheme gives accurate solutions to the pitch-angle scattering equation.

Because of numerical dispersion inherent in approximating the hyperbolic term $-\mu \partial n/\partial x$ of equation (12) with an explicit first-order formulation, the solutions obtained depend slightly on the choice of the $x$ grid size. Numerical stability in the parabolic $\mu$ terms set an upper limit on the time step $\Delta t$ that can be used. At the same time, the accuracy in the hyperbolic $x$ terms is improved by using a larger $\Delta t$ to approximate the characteristics as well as possible (Gerald 1978, § 9.1). The two conditions lead us to use for our $x$ grid size the largest value consistent with reasonable spatial resolution.

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For comparison, the solution to the diffusion equation (5) subject to our boundary and initial conditions can be obtained (e.g., Carslaw and Jaeger 1969), giving

$$U(x, t) = (\text{const.}) \sum_{n=1}^{\infty} \exp \left( -\frac{n^2 \pi^2 t}{6\alpha} \right)$$

$$\times \sin (n\pi x) \sin \left( \frac{n\pi x}{20} \right). \quad (16)$$

We note that for any $x$ the solution to equation (16) depends only on the ratio $t/\alpha$, so it is easy to scale for different parameters. On a plot of $\log U$ versus $\log t$, equation (16) predicts that the shapes of the curves for different $x$'s are all the same. Figure 1 shows the solutions for several values of $x$ at the point $x = 0.2$. Note the strong dependence of the time of maximum and the temporal evolution of $U$ on $x$. For the parameter $x = 1$ to 100, the number of scattering mean free paths between the source and observer is $(\Delta x)/\lambda_x = 0.3\alpha = 0.3$ to 30.

Our numerical solutions to equation (12) give a very different set of intensity-time profiles at $x = 0.2$, as shown in Figure 2. Although the diffusive profile for $x = 10$ is not...
dissimilar to the numerical solution, the diffusive profiles are grossly in error for both the cases $\alpha = 1$ and $\alpha = 100$.

It is easy to understand why the diffusive profile for $\alpha = 1$ is so poor. The mean free path is about 3 times the source-to-observer separation, so the particles are only slightly scattered before being observed. In terms of the discussion of § II, the minimum time $t^*$ for which diffusion can be appropriate is $t^* = \lambda_s = 1/(6\alpha) = 0.17$ for this case. But the diffusive profile has the intensity-time profile entirely passed $x = 0.2$ by this time, so it is clearly inapplicable. Even allowing for the finite spread of the $x$-bins ($\Delta x = 0.05$), rectilinear transit with the particle's speed does not allow any particles to arrive at $x = 0.2$ before $t = 0.05$, by which time the diffusive solution's maximum has already passed. Thus the diffusive solution for $\alpha = 1$ is poor because there is too little scattering between the source and the observer to validate the diffusive approximation.

To understand why the diffusion limit breaks down for strong scattering ($\alpha \geq 10$), it is instructive to consider the particle density in phase space, $\mu$ versus $x$. The diffusion limit assumes that the distribution is nearly isotropic in $\mu$ and simply spreads via spatial diffusion in $x$. As our numerical solutions show, however, the situation is much more complicated. A schematic representation is shown in Figure 3. From an initial anisotropic distribution near the Sun, for strong scattering the particle distribution near the spatial maximum rapidly becomes quasi-isotropic in $\mu$. But the "convective" term $\mu \delta n / \delta x$ in equation (12) causes particles with $\mu > 0$ to be transported quasi-rectilinearly forward in $x$ while those with $\mu < 0$ are transported toward negative $x$. The first particles arriving at any location are those with $\mu \lesssim 1$ that have been most rapidly driven away from $x = x_0$ by the large spatial gradient and the $\mu \delta n / \delta x$ term. By the time of maximum at locations far from $x = x_0$, the pitch-angle distribution has become nearly isotropic. But the distribution does not remain static (i.e., equilibrium has not been reached), since the $\mu \delta n / \delta x$ term causes transport of particles in opposite directions for $\mu < 0$ and $\mu > 0$. At $x = 0.2$, after the maximum has passed, the pitch-angle anisotropy goes through a minimum and then increases again, with the maximum intensity pointing toward the Sun rather than away from it. This is shown in Figure 4, where the late-phase anisotropy is directed back toward $x = 0$. The excess of particles with $\mu < 0$ is due to the backward transport of particles at $x > 0.2$, where the distribution is nearly isotropic.

We emphasize that this complicated interplay between spatial and velocity effects is a direct consequence of the pitch-angle scattering equation (11) or (12). The shape of the pitch-angle scattering coefficient has little influence.

Fig. 3.—Phase-space diagrams of pitch-angle scattering. (a) The $\mu$-term on the right-hand side of eq. (12) gives diffusion of particles in pitch angle. The $\pm$-term represents convection toward positive $x$ for $\mu > 0$ and toward negative $x$ for $\mu < 0$. (b) The initial particle distribution is strongly peaked near the source and for $\mu > 0$. Particles subsequently flow toward positive $x$ and $\mu < 0$. (c) At a later time, with rapid pitch-angle scattering, the distribution is quasi-isotropic in $\mu$. The "convection" term in $x$ gives particle flows as shown.
on the result. Our numerical solutions carried out with a constant pitch-angle scattering coefficient,

$$D_\alpha = 0.8D_0,$$  \hfill (17)

give profiles differing from those shown in Figure 2 by less than 10% for the full range of parameters $1 \leq a \leq 100.$ (The factor of 0.8 in eq. [17] gives the same scattering mean free path as $D_\alpha$ of eq. [11], when calculated using eq. [6].) In our previous work (Gombosi and Owens 1980), we also found similar results for functional forms with $D_\alpha = 0$ at $\mu = 0.$

It should not be surprising that the diffusion limit might fail for nonequilibrium circumstances. Equation (1) has, after all, a diffusive term in velocity but a convective term in the spatial coordinate. The assumption that $\partial n_{11}/\partial t$ is small which enters into the diffusion limit discussed in § II is incorrect for a significant portion of the event in the case of large scattering. Even though the anisotropy may be small for some region of space at a particular time, the particle distribution is rapidly evolving at other points in space. And the distribution function continues to evolve into one with a substantial "equilibrium" anisotropy late in the event.

IV. QUASI-DIFFUSIVE TEMPORAL PROFILES

A fascinating result of our numerical calculations, anticipated by our previous work (Gombosi and Owens 1980), is that the omnidirectional intensity versus time determined by the pitch-angle scattering equation (1) looks very much like the profile obtained from the spatial diffusion equation (5). This is shown strikingly in Figure 5, in which we show the spatial diffusion solution (eq. [16]) with the parameter $\alpha$ arbitrarily varied to "fit" the time of maximum of the numerical solution to equation (12). The "fit" is excellent. On the basis of the intensity-time profile, it would be difficult to distinguish the spatial diffusion result from the pitch-angle scattering integral. But the inferred mean free paths are very different, with the spatial diffusion profile grossly understimating the difference in the amount of scattering between the two "events" shown in the figure.

Thus we obtain the paradoxical result that the solutions to the pitch-angle scattering equation, while establishing the inapplicability of the diffusion limit for the interplanetary propagation of impulsive (i.e., solar flare) events, give intensity-time profiles that look diffusive. The mean free paths inferred from the spatial diffusion equation may, however, be grossly in error. In our models, the numerical integrations give profiles that differ in time-to-maximum ($\tau = wt/L$) by only a factor of 2, even though the scattering mean free path varied by a factor of 100. Such an effect can clearly account for the observed lack of sensitivity of the scattering mean free path with rigidity of solar cosmic rays (e.g., Zwickl and Webber 1977a, b).

![Pitch-Angle Scattering](image1)

Fig. 4.—Pitch-angle distributions. The pitch-angle distributions at $x = 0.2$ are given for the case $\alpha = 10$, for which there are three scattering mean free paths between the source and observer. Time $\tau = 0.1$ corresponds to the first arriving particles, and $\tau = 0.3$ is the time of maximum. By $\tau = 2$, the intensity has decayed to $\sim 20\%$ of its maximum value (see Fig. 2). The source is to the left, and the outer free-escape boundary to the right.

![Spatial Diffusion](image2)

Fig. 5.—"Fitting" of spatial diffusion profiles to those obtained from numerical integration of the pitch-angle scattering equation. The curves are from pitch-angle scattering, with the actual parameters $\alpha = 10$ and 100 shown. The dots and crosses represent spatial diffusion profiles (from eq. [16]), with the parameters $\alpha = 8$ and 12 chosen to give the same time of maximum.
V. DISCUSSION

By direct numerical integration of the pitch-angle scattering equation, we have shown that the diffusion limit is inapplicable to impulsive events like those characterized by impulsive solar flare ejection of cosmic rays into the interplanetary medium. In order to extract information on the scattering mean free path of impulsively injected particles, one must resort to the full pitch-angle scattering equation rather than the spatial diffusion equation derived from it in the quasi-isotropic, quasi-steady-state limit.

A curious result of our numerical calculations is that the intensity-time profiles of the pitch-angle scattering equation for impulsive injection look diffusive. The inferred mean free paths obtained from the diffusion equation, however, can grossly underestimate the actual amount of scattering. Detailed analysis of the pitch-angle distribution (e.g., Bieber et al. 1979) or the anisotropy is thus necessary to deduce useful information concerning the interplanetary transport properties of solar cosmic rays. The intensity-time profiles obtained from the pitch-angle scattering equation are remarkably insensitive to the scattering mean free path, with the time to maximum (in the dimensionless units \( \tau = \nu t/L \)) changing by only a factor of 2 from essentially scatter-free conditions to those with tens of scattering centers between the source and observer.

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T. I. GOMBOSI: Central Research Institute for Physics, H-1525 Budapest, P.O.B. 49, Hungary