Counting Exceptional Curves on Blowups

Takumi Murayama

November 12, 2013

Let $X$ be the blowup of $\mathbb{P}^2$ at $r$ sufficiently general points $p_1, \ldots, p_r$. Repeated applications of [Har77, V, Prop. 3.2] give that $\text{Pic } X \cong \mathbb{Z}^{r+1}$, and is generated by the classes $L, E_1, \ldots, E_r$, where $L$ is (the proper transform of) a line in $\mathbb{P}^2$, and the $E_i$ are the exceptional divisors.

Any $C \neq E_i$ is of the form $C = dL - \sum a_i E_i$ by [Har77, V, Prop. 3.6], where $a_i \geq 0$ are the intersection multiplicities of $C \cap E_i$ and where $d$ is the degree of the image of $C$ in $\mathbb{P}^2$ [Har77, II, Prop. 6.4].

Suppose $C$ is a $(-1)$-curve, i.e., a smooth rational curve with $C^2 = -1$; note that since $C \cong \mathbb{P}^1$, it has genus 0. We first have

$$-1 = C^2 = \left( dL - \sum a_i E_i \right)^2 = d^2 - \sum a_i^2,$$

and so $\sum a_i^2 = d^2 + 1$, by the fact that $L \cdot E_i = 0$ [Har77, V, Prop. 3.2]. By the adjunction formula [Har77, V, Prop. 1.5], we also have

$$-2 = C^2 + C \cdot K_X = -1 + \left( dL - \sum a_i E_i \right) \cdot \left( -3L + \sum E_i \right) = -1 - 3d + \sum a_i,$$

and so $\sum a_i = 3d - 1$.

A priori, we can apply the Cauchy-Schwarz inequality to get

$$\left( \sum a_i \right)^2 \leq r \sum a_i^2 \implies 9d^2 - 6d + 1 \leq rd^2 + r \implies (r - 9)d^2 + 6d + (r - 1) \geq 0.$$ 

For $r = 1$, then, there are no $(-1)$-curves other than $E_1$.

For $r = 2, 3, 4$, $d = 1$ is the only possibility for $(-1)$-curves, i.e., all $(-1)$-curves are lines. Note we have $\sum a_i = \sum a_i^2 = 2$, and so two of the $a_i$ can equal 1.

For $r = 2$, $a_1 = a_2 = 1$, i.e., the only $(-1)$-curve not equal to $E_1, E_2$ is the proper transform of the line that goes through $p_1, p_2$. The dual graph then looks like $\cdot - \cdot - \cdot$, which has automorphism group $\mathbb{Z}/2\mathbb{Z}$. 

1
For $r = 3$, there are $\binom{3}{2}$ possible $(-1)$-curves that are not the $E_i$. The dual graph then is a hexagon, which has automorphism group $D_{12}$.

For $r = 4$, there are $\binom{4}{2}$ possible $(-1)$-curves that are not the $E_i$. To draw the dual graph, we note

$$(L - E_i - E_j)(L - E_k - E_\ell) = 1 - \delta_i^k - \delta_i^\ell - \delta_j^k - \delta_j^\ell,$$

which is 1 if and only if $i \neq j \neq k \neq \ell$. Drawing the dual graph, we end up getting the Petersen graph (which has ten vertices), which has automorphism group $S_5$.

For $r = 5, 6$, we have that $d = 1$ or $d = 2$ are acceptable solutions for $(-1)$-curves, so any $(-1)$-curve that is not an exceptional divisor is either a line or a conic.

For $r = 5$, the $d = 1$ case gives us $\binom{5}{2} = 10$ $(-1)$-curves not equal to the exceptional divisors as above; by the same argument as above, we see that they have intersection 1 if they arise from lines that go through different blown up points. For the $d = 2$ case, we have that $\sum a_i = 5$ and $\sum a_i^2 = 5$, i.e., $a_i = 1$ for all $i$. This conic exists and is unique: it is the unique conic determined by the five points $p_1, \ldots, p_5$. Moreover, we see that the conic has zero intersection with the other $(-1)$-curves that are not the exceptional divisors:

$$(2L - \sum E_i)(L - E_j - E_k) = 2 - 1 - 1 = 0,$$

We therefore have 16 $(-1)$-curves, and the dual graph is a 5-regular graph with 16 vertices, which I don’t know the automorphism group of...

For $r = 6$, the $d = 1$ case gives us $\binom{6}{2} = 15$ $(-1)$-curves that are lines. The $d = 2$ case gives us $\binom{6}{3} = 6$ $(-1)$-curves that are conics. This gives 27 $(-1)$-curves in all. Intersections between two linear $(-1)$-curves are as above. Between a linear and conic $(-1)$-curve, we have

$$\left(2L - \sum_{i \neq j} E_i\right) \cdot (L - E_k - E_\ell) = \delta_i^k + \delta_i^\ell,$$

and so the intersection number is 1 if the blown up point $p_j$ missed by the conic is one of the points which the line goes through. Between two conic $(-1)$-curves,

$$\left(2L - \sum_{i \neq j} E_i\right) \cdot \left(2L - \sum_{i \neq k} E_i\right) = -\delta_j^k.$$
References