THE MINIMAL MODEL PROGRAM

TAKUMI MURAYAMA

ABSTRACT. These notes are taken by Matt Stevenson during the summer mini course on "The Minimal Model Program" from May 30-June 2, 2016. Any and all errors are due to the scribe.

The references for this week, listed from easiest to hardest, are the following:

- Kollár, "The structure of algebraic threefolds."
- Debarré, "Higher-dimensional algebraic geometry."
- Clemens-Kollár-Mori, "Higher dimensional complex geometry."
- Kollár-Mori, "Birational geometry of algebraic varieties."

0. INTRODUCTION

In the 19th century, mathematicians knew how to classify topological (orientable) surfaces by genus g: $g = 0$ was the sphere S^2, $g = 1$ was the torus $\mathbb{T}$, and $g \geq 2$ were the connected sum of tori $\mathbb{T}\# \ldots \#\mathbb{T}$. Alternatively, they can be classified by curvature: S^2 has positive curvature, $\mathbb{T}$ has flat curvature, and higher-genus surfaces have negative curvature.

This also gives a classification of Riemann surfaces: a Riemann surface C of genus $g = 0$ is isomorphic $\mathbf{CP}^1$, one with $g = 1$ is isomorphic to $\mathbf{C}/\mathbf{Z}^2$, and if C has genus $g \geq 2$, then it is isomorphic to $\mathcal{H}/\pi_1(C)$.

Question 0.1. What can we do in higher-dimensions?

Definition 0.2. Two varieties X, Y are *birational* if there are Zariski opens $U \subset X$, $V \subset Y$ such that $U \simeq V$.

It is too difficult to classify isomorphism classes of algebraic varieties in higher-dimensions, so we want instead a way to classify varieties by birational equivalence.

This week we focus on how the presence of rational curves on a variety affects its birational geometry. As an example of this phenomenon, consider the following.

Theorem 0.3 (Castelnuovo)**.** Let X be a smooth proper surface. There exists a nontrivial birational morphism $X \xrightarrow{f} Y$ for some smooth surface Y iff there exists a smooth rational curve $C \subset X$ with $(C^2) = -1$.

We would like to generalize this to higher-dimensions. The idea is that if X has rational curves, it should be "more complicated" and we should be able to simplify it without changing its birational equivalence (as is done in Castelnuovo's theorem by blowing down the curve C).

Conventions. We always work over a field $k = \overline{k}$, and at present we *do not* assume that $\mathrm{char} k = 0$. A variety will mean an integral separated scheme of finite-type over k.

Date: June 1, 2016.

1. Bend & Break Techniques

We want a method to find rational curves on a smooth projective variety. Mori's idea was that certain curves can be degenerated ("bend") into a union of rational curves ("break").

Theorem 1.1 (Mori, '82)**.** Let X be a smooth projective variety and let H be an ample divisor on X. Suppose there is a curve $C' \subset X$ such that $(C' \cdot -K_X) > 0$, then there exists a rational curve $E \subset X$ such that

(1) $\dim X + 1 \geq (E \cdot -K_X) > 0$;
(2)
$$\frac{(E \cdot -K_X)}{(E \cdot H)} \geq \frac{(C' \cdot -K_X)}{(C' \cdot H)}.$$

This theorem will be proven in 2 parts: first we show that C' deforms in some 1-parameter family, and second we show that this family degenerates into rational curves.

1.1. Break Lemmas.

Theorem 1.2 (Deformation Lemma)**.** Let X be a proper variety, let C be a smooth projective curve, let $p \in C$, and let $g_0 \colon C \to X$ be a nonconstant morphism. Suppose there is a nontrivial 1-parameter family of morphisms $g_t \colon C \to X$ parametrized by $t \in D_0$, where D_0 is a (not necessarily proper) curve, such that $g_0(p) = g_t(p)$ for all $t \in D_0$. Then, X contains a rational curve through $g_0(p)$.

This is the main technical tool used to get rational curves.

Lemma 1.3 (Rigidity Lemma)**.** Let $Y \xrightarrow{f} Z$ be a proper surjective with connected fibres of dimension n and let $Y \xrightarrow{g} X$ be another morphism. If $g(f^{-1}(z_0))$ is a point for some $z_0 \in Z$, then $g(f^{-1}(z))$ is a point for all $z \in Z$.

Proof. Set $W := \operatorname{im}(h := f \times g) \subset Z \times X$, so we get a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{h} & W & & \\ & \searrow^{f} & \downarrow^{p} & \searrow^{q} & \\ & & Z & & X \end{array}$$

where $p.q$ are the projections . Both h and p are proper. Then, $p^{-1}(z) = h(f^{-1}(z))$ and $\dim p^{-1}(z_0) = 0$, by assumption. By the upper semicontinuity of fibre dimension, there is an open $z_0 \in U \subset Z$ such that $\dim p^{-1}(z) = 0$ for all $z \in U$. Then h has fibre dimension n over $p^{-1}(U)$, so h has fibre dimension $\geq n$ everywhere (by upper semicontinuity).

Now, for all $w \in W$, $h^{-1}(w) \subset f^{-1}(p(w))$ and $\dim h^{-1}(w) \geq n$ and $\dim f^{-1}(p(w)) = n$, so $h^{-1}(w)$ is a union of irreducible components of $f^{-1}(p(w))$. Thus, $h(f^{-1}(p(w))) = p^{-1}(p(w))$ is finite; in fact, it is a single point, since $f^{-1}(p(w))$ is connected. Now,
$$g(f^{-1}(z)) = q(h(f^{-1}(p(w))))$$
is a point, where we choose $z = p(w)$. □

Proof of Deformation Lemma. Compactify $D_0 \hookrightarrow D$, where D is a proper curve, so we have a rational map $g \colon C \times D \dashrightarrow X$. Assume that g is regular and defined everywhere, then $g(\{p\} \times D)$ is a point, by assumption. The Rigidity Lemma implies that $g(\{c\} \times D)$ is a point for all $c \in C$, so g_t is a trivial family, which is a contradiction.

Therefore, we know that g is not defined somewhere along $\{p\} \times D$. Let
$$Z_n \to Z_{n-1} \ldots \to C \times D$$
be a minimal sequence of blowups resolving the indeterminacies of the rational map $g \colon C \times D \dashrightarrow X$, so we a morphism $Z_n \xrightarrow{f} X$. Let $E \subset Z_n$ be the exceptional divisor of the last blowup.

We want to show that $f(E)$ is a rational curve. As $E \simeq \mathbf{P}^1$, the only way that $f(E)$ is not birational to $\mathbf{P}^1$ is if $f(E)$ is a point. However, if $f(E)$ is a point, then the last blowup $Z_n \to Z_{n-1}$ would not be necessary. Therefore, $f(E)$ is birational to $\mathbf{P}^1$ and $g_0(p) \in f(E)$, as f arose by blowing up centres along $\{p\} \times D$. □

In the context of Theorem 1.1, the Deformation Lemma says that if C' bends, then it breaks.

1.2. **Bend Lemmas & Proof of the Theorem.** We proceed in several steps.

Step 1. Let $f\colon C \to C' \hookrightarrow X$ be the normalization of C', then we want to deform f. A deformation of C looks like a section of $H^0(C, f^*T_X)$. This is made precise by the following fact from deformation theory.

Fact 1.4. The dimension of deformations of $f\colon C \to X$ is bounded below by $h^0(C, f^*T_X) - h^1(C, f^*T_X)$.

See Chapter 2 of Debarré for a proof of the above fact. Using Riemann-Roch for vector bundles, we see that

$$h^0(C, f^*T_X) - h^1(C, f^*T_X) = \deg f^*T_X + (1 - g(C))\operatorname{rank}(T_X) = (f_*C \cdot -K_X) + (1 - g(C))\dim X,$$

where the last equality follows from the projection formula. The quantity $(f_*C \cdot -K_X)$ is assumed to be positive in the statement of Theorem 1.1. We want to keep the image of $p \in C$ fixed so that the deformation space has dimension greater than or equal to

$$(f_*C \cdot -K_X) - g(C)\dim X \tag{$*$}$$

As long as this number is positive, we get deformations and can use the Deformation Lemma.

Step 2. Let's make this number positive.

- If $g(C) = 0$, then $(*)$ is positive by hypothesis, so we can deform (but this is not needed since the curve C' is already rational).
- If $g(C) = 1$, then let $C \xrightarrow{n} C$ be the "multiplication-by-n" morphism, so
$$((f \circ n)_*(C) \cdot -K_X) = n^2(f_*C \cdot -K_X),$$
since "multiplication-by-n" has degree n^2. Then, $(*)$ is eventually positive, as $g(C)$ doesn't change.
- If $g(C) \geq 2$, then C has no endomorphisms of degree > 1, by Hurwitz. Similarly, sheeted covers of C don't work.

Question 1.5. How can we get self-maps of C?

The solution is to pass to char$k > 0$ and use Frobenius! Suppose char$k = p$, then

$$\big((f \circ F_p^m)_*(C) \cdot -K_X\big) = p^m\,(f_*C \cdot -K_X)\,,$$

where F_p denotes the Frobenius morphism in characteristic p. Then, $(*)$ is positive for $m \gg 0$.

Step 3. Reducing mod p.

Embed $C' \subset X \subset \mathbf{P}^N$, then C' and X are defined by polynomials over $R = \mathbf{Z}[a_i]$, a finite algebra extension of $\mathbf{Z}$ viewed as a subring of k (e.g. take the a_i's to be the coefficients of the polynomials cutting out C' and X in $\mathbf{P}^N$). By enlarging R, we can assume $f\colon C \to X$ is also defined over R. Therefore, we get the diagram

$$\begin{array}{ccccc} \mathfrak{X} & \hookleftarrow & \mathfrak{X}_0 & \longleftarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} R & \hookleftarrow & \operatorname{Spec} K & \longleftarrow & \operatorname{Spec} k \end{array}$$

where $K = \operatorname{Frac}(R)$ and $k = \overline{K}$. By generic flatness/freeness, the quantities $((f_\mathfrak{p})_*(C_\mathfrak{p}) \cdot -K_{X_\mathfrak{p}})$, $g(C_\mathfrak{p})$, and $\chi(f_\mathfrak{p}^*(C_\mathfrak{p}))$ are constant as a function of $\mathfrak{p} \in U$, for some open set $U \subset \operatorname{Spec} R$ (because they are all defined by cohomological data which is constant in flat families).

Let $p = \mathrm{char} R/\mathfrak{p}$, then the Frobenius F_p acts on $C_\mathfrak{p}$, so we can compose

$$C_\mathfrak{p} \xrightarrow{F_p^m} C_\mathfrak{p} \xrightarrow{f} X_\mathfrak{p},$$

and hence

$$(*) = p^m \left((f_\mathfrak{p})_*(C_\mathfrak{p}) \cdot -K_{X_\mathfrak{p}}\right) - g(C_\mathfrak{p}) \dim X_\mathfrak{p}$$

is positive for $m \gg 0$. The Deformation Lemma then implies that the cycle $p^m f_*(C_\mathfrak{p})$ is algebraically equivalent to $f_m^1(C_\mathfrak{p}) + Z_{p,m}^1$, where $Z_{p,m}^1$ is some algebraic cycle equal to a sum of rational curves and $f_m^1(C_\mathfrak{p})$ is the rest. So far, we have shown the theorem for the fibres $\mathfrak{X}_\mathfrak{p}$ for $\mathfrak{p} \in U \subset \mathrm{Spec}\ R$ (or at least we know rational curves exist).

Step 4. Ensure that inequality (2) holds. Set

$$M := \frac{(C' \cdot -K_X)}{(C' \cdot H)},$$

then our goal is to show that $\frac{(E \cdot -K_X)}{(E \cdot H)} > M - \epsilon$ for all $\epsilon > 0$. We make two observations:

- By generic freeness, $\frac{(f_*(C_\mathfrak{p}) \cdot -K_{X_\mathfrak{p}})}{(f_*(C_\mathfrak{p}) \cdot H_\mathfrak{p})} = M$ for all $\mathfrak{p} \in U$.
- This ratio M does not change under precomposition with the Frobenius.

We claim that $f_m^1(C_\mathfrak{p})$ satisfies

$$(f_m^1(C_\mathfrak{p}) \cdot -K_{X_\mathfrak{p}}) \le g(C_\mathfrak{p}) \dim X_\mathfrak{p}.$$

If not, then $(f_m^s(C_\mathfrak{p}) \cdot -K_{X_\mathfrak{p}}) - g(C_\mathfrak{p}) \dim X_\mathfrak{p} > 0$, so by the Deformation Lemma,

$$f_m^s(C_\mathfrak{p}) + Z_{\mathfrak{p},m}^s \sim f_m^{s+1}(C_\mathfrak{p}) + Z_{\mathfrak{p},m}^{s+1}$$

and we can repeat this procedure until the claim is satisfied. Let

$$a = (f_m^s(C_\mathfrak{p}) \cdot -K_{X_\mathfrak{p}}),$$

$$b = (Z_{\mathfrak{p},m}^s \cdot -K_{X_\mathfrak{p}}),$$

$$c = (f_m^s(C_\mathfrak{p}) \cdot H_\mathfrak{p}),$$

$$d = (Z_{\mathfrak{p},m}^s \cdot H_\mathfrak{p}).$$

Then, $M = \frac{a+b}{c+d}$.

Claim. For any $\epsilon > 0$ and $m \gg 0$, there is an irreducible component $E_\mathfrak{p}$ of $Z_{\mathfrak{p},m}^s$ such that

$$\frac{(E_\mathfrak{p} \cdot -K_{X_\mathfrak{p}})}{(E_\mathfrak{p} \cdot H_\mathfrak{p})} > M - \epsilon.$$

We first require the following lemma.

Lemma 1.6. If $c, d > 0$, then $\frac{a+b}{c+d} \le \max\left\{\frac{a}{c}, \frac{b}{d}\right\}$.

Proof of Claim. There are 2 cases to consider: when c gets large for $m \gg 0$ or when c stays bounded. Note that, for large m, $c + d$ gets large, so $a + b$ must get large. But by the previous claim, a is bounded, so b must get large.

Case 1: if c gets large, then $\frac{a}{c} < M$ for $m \gg 0$, so $\frac{b}{d} \ge M$. From Lemma 1.6, we can split off the requisite $E_\mathfrak{p}$.

Case 2: if c stays bounded, then b and d get large, so $\frac{b}{d} + \epsilon > \frac{a+b}{c+d} = M$. For $m \gg 0$, $\frac{b}{d} > M - \epsilon$, so one again $E_\mathfrak{p}$ exists. □

Step 5. $E_\mathfrak{p}$ satisfies inequality (1). If this is not the case, then $(E_\mathfrak{p} \cdot -K_{X_\mathfrak{p}}) > \dim X + 1$, so by the Deformation Lemma,

$$(E_\mathfrak{p} \cdot -K_{X_\mathfrak{p}}) + \dim X_\mathfrak{p} > 2\dim X_\mathfrak{p} + 1 > 0,$$

so $E_\mathfrak{p}$ moves in a ≥ 2-dimensional family; in particular, $E_\mathfrak{p}$ moves in a 1-dimensional family. Repeat this and then the number $(E_\mathfrak{p} \cdot -K_{X_\mathfrak{p}})$ goes down.

Step 6 Lift to characteristic zero. Observe that $(E_\mathfrak{p} \cdot H_\mathfrak{p}) \leq \frac{\dim X+1}{M}$, which is independent of $\mathfrak{p}$. Thus, these curves $E_\mathfrak{p}$ all have bounded degree when embedding in $X \subset \mathbf{P}^N$, using H.

Proposition 1.7. Let R be a finite $\mathbf{Z}$-algebra. If a homogeneous system of algebraic equations with R-coefficients has a nontrivial solution in $\overline{R/\mathfrak{p}}$ for $\mathfrak{p}$ in a Zariski open set $U \subset \operatorname{Spec} R$, then the equations also have a solution over $\overline{\operatorname{Frac} R}$.

Proof. The equations define a closed subscheme $Z \subset \mathbf{P}^N_{\operatorname{Spec} R}$. The projection $\pi\colon \mathbf{P}^N_{\operatorname{Spec} R} \to \operatorname{Spec} R$ is proper, so $\pi(Z)$ is closed. But, we know that $\pi(Z) \supset U$, so $\overline{U} = \operatorname{Spec} R = \pi(Z)$. □

For us, we want rational curves $E \subset X$, i.e. we want maps $\mathbf{P}^1 \to X \subset \mathbf{P}^N$, given by $[t_0\colon t_1] \mapsto [g_{\mathfrak{p}_0}\colon \ldots \colon g_{\mathfrak{p}_n}]$ a tuple of homogeneous forms. We claim that we can make

$$\deg g_{\mathfrak{p}_i} \leq \frac{\dim X + 1}{M}.$$

Let $h_j(x_0, \ldots, x_n)$ be the homogeneous polynomials defining X (which have coefficients in R), then having a morphism $\mathbf{P}^1 \to X$ of this form is equivalent to a set of equations with variables being coefficients of the $g_{\mathfrak{p}_i}$'s. By Proposition 1.7, we know solutions exist for all $\mathfrak{p} \in U \subset \operatorname{Spec} R$, so we get solutions over the generic point.

This concludes the proof of bend and break (though we have not shown that the lifts to characteristic zero also satisfy the inequalities (1) and (2)).

Remark 1.8.
- Mori's motivation for bend and break was to prove Hartshorne's conjecture, which states that T_X is ample iff $X \simeq \mathbf{P}^n_k$, but he needed that Fano varieties are covered by rational curves (i.e. each point has a rational curve through it).
- With this technique, Kollár-Miyaoka-Mori ('92) showed that Fano varieties are rationally connected (that is, any 2 points have a rational curve passing through them).
- Using bend and break, Boucksom-Demailly-Páun-Peternell showed that K_X is not pseudoeffective iff X is uniruled.

2. Cones of Curves

The key idea of Hironaka, which inspired much of what we study below, is that maps of projective varieties are in fact determined by what they do to curves (that is, by which curves the maps contract). We will package all of this information into one geometric object, which we call the Mori cone.

Definition 2.1. A subset $N \subseteq \mathbf{R}^n$ is a *cone* if $0 \in N$ and N is closed under multiplication by $\mathbf{R}_{>0}$. A subcone $M \subset N$ is *extremal* if for $u, v \in N$, $u + v \in M$ implies that $u, v \in M$; that is, "N lies on one side of M". The cone N is *convex* if $u, v \in N$ implies that $u + v \in N$.

Definition 2.2. The *cone of effective curves* on X is

$$\operatorname{NE}(X) = \{\sum_i a_i[C_i]\colon C_i \subset X \text{ curve, } a_i \in \mathbf{R}_{\geq 0} \subset N_1(X),$$

where $N_1(X)$ is the Néron-Severi group (which consists of algebraic 1-cycles on X modulo numerical equivalence). For $X/\mathbf{C}$, $\operatorname{NE}(X)$ is a subset of $H_2(X, \mathbf{R})$. The *Mori cone* is defined to be the closure of $\operatorname{NE}(X)$, and it is denoted $\overline{\operatorname{NE}}(X)$. The Mori cone is convex.

The "simplified" version of the cone theorem for smooth varieties is given below.

Theorem 2.3 (Mori, '82)**.** Let X be a nonsingular projective variety. There are countably-many rational curves $C_i \subset X$ such that

$$0 < (C_i \cdot -K_X) \leq \dim X + 1$$

and so that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i \mathbf{R}_{\geq 0} \cdot [C_i],$$

where $\overline{\mathrm{NE}}(X)_{K_X \geq 0}$ is the subset of $\overline{\mathrm{NE}}(X)$ consisting of those curves C such that $(C \cdot K_X) \geq 0$.

2.1. Examples.

Lemma 2.4. Let X be a smooth projective surface. Then,

(1) $(C^2) \leq 0$ implies that $[C] \in \partial\overline{\mathrm{NE}}(X)$.
(2) $(C^2) < 0$ implies that $[C]$ is extremal.

Proof. Let D be an irreducible curve in X. If $(D \cdot C) < 0$, then $D = C$, so

$$\overline{\mathrm{NE}}(X) = \mathbf{R}_{\geq 0} \cdot [C] + \overline{\mathrm{NE}}(X)_{C \geq 0}.$$

This gives (2). □

Example 2.5. $\mathbf{P}^2$, thought of analytically, has $H_2(\mathbf{P}^2, \mathbf{R}) = \mathbf{R}$; thus, the Mori cone is the ray $\mathbf{R}_{\geq 0}$ and it is generated by the class of a line ℓ.

Example 2.6. The Mori cone of $\mathrm{Bl}_p\mathbf{P}^2$ is generated by the exceptional divisor E and $\ell - E$, which is the strict transform of a line through the point p.

Example 2.7. For $X = \mathrm{Bl}_{p_1,\ldots,p_{10}}\mathbf{P}^2$, the Cone Theorem implies that the Mori cone decomposes into 3 "pieces": where $(K_X \cdot -) < 0$, the boundary consists of a countable number of extremal rays. There are also the pieces where $(K_X \cdot -) = 0$ and $(K_X \cdot -) > 0$, but no one knows what happens in the latter.

2.2. Proof of the Cone Theorem.

Consider the subset

$$\{C \in \mathrm{NE}(X) \colon 0 < (C \cdot -K_X) \leq \dim X + 1 \text{ and the coefficients of } C \text{ are in } \mathbf{Q}\},$$

then this is countable (since it consists of certain points with rational coordinates in the finite-dimensional real vector space $N_1(X)$), and these will be our C_i's. Set

$$W := \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i \mathbf{R}_{\geq 0} \cdot [C_i],$$

then we would like to show that $\overline{W} = \overline{\mathrm{NE}}(X)$. To show that W is already a closed cone involves some convex geometry, which we will not discuss here.

The inclusion $\overline{W} \subseteq \overline{\mathrm{NE}}(X)$ is trivial. To show the opposite inclusion, the idea is to argue by contradiction and suppose that we can find a divisor M which is positive on W, but zero somewhere on $\overline{\mathrm{NE}}(X)$, say at Z. We can approximate M by an ample and Z by an effective and then apply bend and break.

Step 1. Find a suitable ample divisor.

First we claim that there is a divisor D such that $(D \cdot -)$ is positive on $\overline{W}\backslash\{0\}$, and $(D \cdot -)$ is negative somewhere on $\overline{\mathrm{NE}}(X)$. To see this, we need the following fact.

Fact 2.8. The dual of $\overline{\mathrm{NE}}(X)$ is $\mathrm{Nef}(X)$.

If $W \subsetneq \overline{\mathrm{NE}}(X)$, then $\overline{W}^* \supsetneq \mathrm{Nef}(X)$, so we can take $D \in \overline{W}^* \backslash \mathrm{Nef}(X)$. Take an ample divisor H, then we will use $M = H + \mu D$ lies on the boundary of the nef cone, for some $\mu \in \mathbf{R}_{\geq 0}$. We will use $H + \mu' D$ as our ample divisor, for $\mu' < \mu$.

Step 2. Find a suitable effective divisor.

By assumption, $(Z \cdot (H + \mu D)) = 0$ for some $Z \in \overline{\mathrm{NE}}(X) \backslash \overline{W}$ and also $(Z \cdot K_X) < 0$, since otherwise $\overline{\mathrm{NE}}(X)_{K_X \geq 0} \subset \overline{W}$. Pick a sequence $Z_k = \sum_j a_{k_j} Z_{k_j} \in \mathrm{NE}(X)$ such that $Z_k \to Z$ as $k \to \infty$. By Lemma 1.6,

$$\max_j \frac{(Z_{k_j} \cdot -K_X)}{(Z_{k_j} \cdot (H + \mu' D))} \geq \frac{(Z_k \cdot -K_X)}{(Z_k + (H + \mu' D))},$$

and say they are achieved by Z_{k_0}.

Step 3. Apply bend and break to Z_{k_0} and $H + \mu' D$ to get rational curves $E_{i(k)}$ such that

$$\dim X + 1 \geq (E_{i(k)} \cdot -K_X) > 0$$

and so that

$$\frac{(Z_k \cdot -K_X)}{(Z_k + (H + \mu' D))} \leq \frac{(Z_{k0} \cdot -K_X)}{(Z_{k0} + (H + \mu' D))} \leq \frac{(E_{i(k)} \cdot -K_X)}{(E_{i(k)} \cdot (H + \mu' D))} \leq \frac{(E_{i(k)} \cdot -K_X)}{(E_{i(k)} \cdot H)} \leq M,$$

where the second-to-last inequality follows since $E_{i(k)} \equiv_{\mathrm{num}} C_i$ for some i, so $E_{i(k)} \in W$ and $(E_{i(k)} \cdot D) \geq 0$. The constant $M \gg 0$ is chosen so that $MH + K_X$ is ample, in which case $(E_{i(k)} \cdot (MH + K_X)) \geq 0$, so

$$M \geq \frac{(E_{i(k)} \cdot -K_X)}{(E_{i(k)} \cdot H)}.$$

Sending $k \to \infty$ and $\mu' \to \mu$ we get that $(Z \cdot (H + \mu D)) = 0$, which is a contradiction. Therefore, $\overline{W} = \overline{\mathrm{NE}}(X)$. This completes our proof of the cone theorem.

3. Contractions Theorems and the MMP

We now know something about the structure of $\overline{\mathrm{NE}}(X)$ and we want to use it to produce "contraction morphisms".

Theorem 3.1 (Contraction Theorem)**.** Let $F \subset \overline{\mathrm{NE}}(X)$ be an extremal face lying in $\overline{\mathrm{NE}}(X)_{K_X < 0}$, then there is a contraction morphism $\mathrm{cont}_F \colon X \to Y$ such that $\mathrm{cont}_F(C)$ is a point iff $[C] \in F$.

The proof of this result is quite involved and it will not be discussed. The basic idea for surfaces is to use Castelnuovo's theorem. There are 3 possible cases for contraction morphisms produced by the theorem:

(1) $\dim X > \dim Y$;
(2) $\dim X = \dim Y$ (e.g. a blowup) and either:
   2.1. (Divisorial Contractions) the exceptional locus $\mathrm{exc}(\mathrm{cont}_F)$ has $\mathrm{codim}\,(\mathrm{exc}(\mathrm{cont}_F)) = 1$.
   2.2. (Small Contractions) $\mathrm{codim}\,(\mathrm{exc}(\mathrm{cont}_F)) \geq 2$.

3.1. **MMP for Surfaces.**

For surfaces, small contractions don't exist. The Contraction Theorem in this context is the following.

Theorem 3.2. Let X be a smooth projective surface and let $R \subset \overline{\mathrm{NE}}(X)_{K_X < 0}$ be an extremal ray (i.e. it is extremal and 1-dimensional), then $\mathrm{cont}_R \colon X \to Z$ exists and it is one of the following types:

(1) Z is a smooth surface, $X \to Z$ is the blowup at a point, and $\dim N_1(X) > \dim N_1(Z)$.
(2) Z is a smooth curve and X is a minimal ruled surface (the contraction map is the structure map for the ruled surface).
(3) Z is a point and in fact $X \simeq \mathbf{P}^2$.

We can repeat this process again and again so that the surface becomes simpler and simpler; the process terminates because $\dim N_1(X) < \infty$ and blowing down a point decreases the dimension. The end result is that we are either in case (2) or (3), or K_X becomes nef and there are no more extremal rays to contract (since $\overline{\mathrm{NE}}(X)_{K_X<0} = \emptyset$).

3.2. **Threefolds.** The situation in higher-dimensions is considerably more complicated, as the case of 3-folds illustrates.

Theorem 3.3. Let X be a smooth projective 3-fold over $\mathbf{C}$, then $\mathrm{cont}_R\colon X \to Y$ exists as before and it is one of the following types:

(1) (C) cont_R is a conic bundle, i.e. $\dim Y = 2$ and the fibres are conics;
(2) (D) cont_R is a del Pezzo fibration, i.e. $\dim Y = 1$ and the fibres are del Pezzo surfaces;
(3) (F) $\dim Y = 0$ and X is a Fano variety;
(4) (E) The exceptional cases: $\dim Y = 3$, cont_R is birational, and one of the following holds:
   4.1. (E1) cont_R is the blowup of a curve;
   4.2. (E2) cont_R is the blowup of a point;
   4.3. (E3-E5) Y is singular, and one of the following holds:
      (i) (E3) Y is the blowup of a double point, i.e. it is locally
$$\mathbf{C}[[x,y,z,w]]/(x^2+y^2+z^2+w^2);$$
      (ii) (E4) Y is the blowup of a singular point of the form
$$\mathbf{C}[[x,y,z,w]]/(x^2+y^2+z^2+w^3);$$
      (iii) (E5) cont_R contract a projective plane $\mathbf{P}^2$ with normal bundle $\mathcal{O}(-2)$, i.e. it is locally
$$\mathbf{C}[[x^2,y^2,z^2,xy,yz,zw]]$$

The issue is that for 3-folds we cannot just repeatedly apply the Cone and Contraction Theorems, since these theorems (or at least our proofs) only work for smooth varieties. We therefore need to enlarge our category of varieties to include some singular ones.

To decide which singular ones we want, we make the following observations:

(1) the class should be normal (so that we have Weil divisors);
(2) K_X should be ($\mathbf{Q}$-)Cartier, so that intersecting K_X with a curve makes sense (even better, we could ask that X be $\mathbf{Q}$-factorial; that is, all Weil divisors are $\mathbf{Q}$-Cartier).

It looks like we need Cone and Contractions theorems for arbitrary normal $\mathbf{Q}$-factorial varieties. This presents another issue, as (2) does not behave well with respect to small contractions.

Proposition 3.4. Let X be a normal $\mathbf{Q}$-factorial variety and let $f\colon X \to Y$ be a small contraction, then K_Y is not $\mathbf{Q}$-Cartier.

Proof. Suppose K_Y is $\mathbf{Q}$-Cartier, so that mK_Y and mK_X are Cartier for m divisible enough. Then, $f^*(mK_Y)$ and mK_X are still Cartier. Moreover, mK_X and $f^*(mK_Y)$ are linearly equivalent outside of the exceptional locus of f. As X is normal, Hartog's lemma implies that mK_X and mf^*K_Y are in fact linearly equivalent, which is a contradiction: if $[C]$ is a curve class which is contracted, then
$$0 = (f_*C \cdot mK_Y) = (C \cdot f^*mK_Y) = (C \cdot mK_X) < 0.$$
□

3.3. **Flips.** The notion of flips provide a way to use the Contraction Theorem, even when small contractions can occur. The idea is the following: if $(K_X \cdot C) < 0$, then we "cut out" the curve C and replace it with another curve C^+ such that $(K_X \cdot C^+) > 0$ and so that it cannot be contracted any further.

Example 3.5. Let's start backwards and describe the end result of the flip: consider a 3-fold X^+ containing a rational curve C^+ such that $(K_{X^+} \cdot C^+) > 0$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$ (e.g. take

$$X^+ = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(2)),$$

and C^+ is the section corresponding to $\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(2) \to \mathcal{O}_{\mathbf{P}^1} \to 0$), and consider the morphism $X^+ \to Y$ given by contracting C^+. Construct $X_1 \to X^+$ by blowing up C^+, giving an exceptional divisor $E_1 \subset X_1$; construct $X_2 \to X_1$ by blowing up E_1; this gives a copy of $\mathbf{P}^1 \times \mathbf{P}^1 \subset X_2$, from which we build X_3 with a copy of $\mathbf{P}^2$ intersecting a copy of $\mathbf{P}^1$. Contract the $\mathbf{P}^2$ to a curve C get X, then $(C \cdot K_X) < 0$.

This fixes the issue: we no longer have the Contraction Theorem telling us that C^+ can be contracted. See Debarré's book for the details of this example.

Definition 3.6. Let $f : X \to Y$ be a proper birational morphism with $\operatorname{codim}(\operatorname{exc}(f)) \geq 2$ in X, K_X is $\mathbf{Q}$-Cartier, and $-K_X$ is f-ample (in particular, the intersection with a contracted curve is positive). Then, a variety X^+ with a proper birational morphism $f^+ : X^+ \to Y$ is called a *flip* provided

(1) K_{X^+} is $\mathbf{Q}$-Cartier;
(2) K_{X^+} is f^+-ample;
(3) $\operatorname{codim}(\operatorname{exc}(f^+)) \geq 2$ in X^+.

In particular, we have the following commutative diagram:

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^+ \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f^+} & \\ Y & & \end{array}$$

Question 3.7. Do flips exist?

In '04, Hacon-McKernan showed that flips do exist, given certain singularity assumptions on X.

Question 3.8. Do flips terminate?

This is true in dimension-3 (due to Reid-Shokurov), but it is open in higher dimensions.

4. Generalizations to the singular setting

We can no longer use bend and break methods if X is singular, so our proofs of the Cone and Contraction Theorems do not hold. The idea will be to prove things "in reverse": we show the Nonvanishing Theorem, then the Basepoint-Free Theorem, the Contraction Theorem, and finally the Cone Theorem.

4.1. **Preliminaries.** We would like a class of singularities in which the MMP stays and works well; this will be the class of klt singularities.

Definition 4.1. (1) A *(log) pair* (X, Δ) consists of a normal variety X and a $\mathbf{Q}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbf{Q}$-Cartier.

(2) Let $f : Y \to X$ be a log resolution of a log pair (X, Δ) (that is, a proper birational morphism with Y smooth and $f^*(K_X + \Delta)$, then we may write

$$K_Y \sim f^*(K_X + \Delta) + \sum_E a(E, \Delta) \cdot E,$$

where the sum runs over the (prime) exceptional divisors of f. The coefficient $a(E, \Delta)$ is called the *(log) discrepancy* of E with respect to (X, Δ).

(3) Assuming $\Delta = \sum_i d_i D_i$ for $d_i \in \mathbf{Q} \cap [0,1]$ and D_i prime, we say that a log pair (X, Δ) is

$$\begin{cases} \text{klt} \\ \text{lc} \end{cases} \quad \text{if for all exceptional divisors } E \text{ over } X,\ a(E,\Delta) \begin{cases} > -1 \text{ and } \lfloor \Delta \rfloor = 0 \\ \geq -1. \end{cases}$$

These abbreviations stand for Kawamata log terminal (klt) and log canonical (lc).

Remark 4.2. A few comments on why we care about pairs:

- Log pairs are useful to study open varieties: if X^0 is a non-proper variety, take X to be a compactification and Δ to be the boundary, then (X, Δ) is a log pair. Iitaka showed that, using (X, Δ), one still has well-defined de Rham cohomology groups for X^0, independent of the choice of compactification.
- Log pairs offer more flexibility: there is more room to change Δ without worrying about K_X not being $\mathbf{Q}$-Cartier, or even to avoid issues like $K_X \equiv_{\text{num}} 0$.
- Inductive proofs: in our applications, we will often modify Δ so as to be able to run an induction argument.

4.2. Statements of the Theorems.

Theorem 4.3 (Nonvanishing Theorem)**.** Let (X, Δ) be a klt pair and let L be a nef Cartier divisor on X. Suppose there is $p > 0$ such that $pL - (K_X + \Delta)$ is big and nef, then the linear system $|nL|$ is nonempty for all $n \gg 0$.

Theorem 4.4 (Basepoint-Free Theorem)**.** Under the assumptions of Theorem 4.3, the linear system $|nL|$ is basepoint-free for all $n \gg 0$.