Log Canonical Thresholds and Valuations

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Abstract

The log canonical threshold is an invariant of singularities in algebraic geometry. Given a polynomial f in n variables such that $f(0) = 0$, the log canonical threshold of f at the origin is the supremum over all real numbers c such that $|1/f|^c$ is L^2 at the origin. Thus, this invariant measures the (possible) singularity of the hypersurface $\{f = 0\}$ at the origin. While the invariant was first studied from the analytic viewpoint in as far back as the 1950s, it currently receives considerable interest in the field of birational geometry. During this three days course we will discuss

- Basic properties of the log canonical threshold,
- Generic limits and the ACC Conjecture, and
- The space of $\mathbf{R}$-valued valuations over a variety.

Reference:

János Kollár. *Which powers of holomorphic functions are integrable?* May 6, 2008. arXiv: `0805.0756 [math.AG]`.

Contents

*Notes were taken by Takumi Murayama, who is responsible for any and all errors.

1 Introduction

Lecture 1

We will be talking about log canonical thresholds, which measure how bad singularities are. The structure of the course is as follows:

- Basic properties of the log canonical threshold,
- Generic limits and the ACC Conjecture (and partial proof, which is very exciting and beautiful), and
- How log canonical thresholds arise as a minimum on the space of $\mathbf{R}$-valued valuations over a variety.

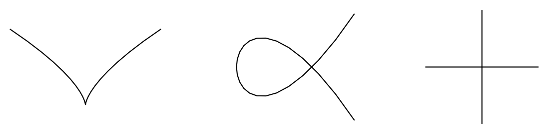
1.1 Setup

Let $f \in \mathbf{C}[z_1, \ldots, z_n]$ such that $f(0) = 0$. We want to study the singularities of the hypersurface $\{f = 0\} \subset \mathbf{A}^n$ at the origin 0.

Recall the *multiplicity* $\operatorname{ord}_0(f)$, defined to be the largest power the maximal ideal f is in, or alternatively the lowest degree term that exists in f.

- If $\operatorname{ord}_0(f) = 1$, then $\{f = 0\}$ is smooth at 0;
- If $\operatorname{ord}_0(f) > 1$, then we have a singularity.

Examples 1.1. The following all have the same order of vanishing:



$x^2 - y^3 = 0$ $\quad$ $y^2 - x^3 - x^2 = 0$ $\quad$ $xy = 0$

Log canonical thresholds an distinguish them, however.

The order of vanishing can be thought of as using a particular valuation to measure the singularity, but we will see on Thursday that the log canonical threshold can be thought of as using information from all valuations.

Today, we will define the log canonical threshold analytically.

Definition 1.2. The *log canonical threshold* of f at 0 is the real number $\operatorname{lct}_0(f)$ such that

- $\frac{1}{|f|^\lambda}$ is L^2 in the neighborhood of 0 for $\lambda < \operatorname{lct}_0(f)$;
- $\frac{1}{|f|^\lambda}$ is L^2 in the neighborhood of 0 for $\lambda > \operatorname{lct}_0(f)$.

We also define $\operatorname{lct}(f) = \min_{p \in \{f=0\}} \operatorname{lct}_p(f)$.

We will see that $\operatorname{lct}_0(f)$ exists, and that $\operatorname{lct}_0(f) \in \mathbf{Q}_{>0}$.

Example 1.3. Let $f = z^a \in \mathbf{C}[z]$. We want to investigate when

$$\int \frac{1}{|z^a|^{2\lambda}} < \infty$$

in a neighborhood of 0. We use polar coordinates:

$$\int \frac{1}{|z^a|^{2\lambda}} = \int_0^{2\pi} \int_0^{\epsilon} \frac{1}{\rho^{2\lambda a}} \rho \, d\rho \, d\theta = 2\pi \int_0^{\epsilon} \frac{1}{\rho^{2a\lambda - 1}} \, d\rho < \infty$$

which holds if and only if $2a\lambda - 1 < 1$, i.e., $\lambda < 1/a$. Thus, $\operatorname{lct}_0(z^a) = 1/a$. Moreover,

- By Fubini's theorem, this also implies

$$f = z_1^{a_1} \cdots z_n^{a_n} \in \mathbf{C}[z_1, \ldots, z_n]$$

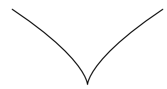
  has $\operatorname{lct}_0(f) = \min_i\{1/a_i\}$;
- If $\{f = 0\}$ is smooth at 0, then $f = uz$ where u is locally invertible, and so $\operatorname{lct}_0(f) = 1$.

The following says that the case where $\{f = 0\}$ is smooth at 0 gives the "best possible" log canonical threshold.

Claim 1.4. $\mathrm{lct}_0(f) \leq 1$.

Proof. Note for $\epsilon > 0$, there always exists $p \in B_0(\epsilon) \cap \{f = 0\}$ such that $\{f = 0\}$ is smooth at p. This implies that at p, $f = uz^m$ where u is locally invertible and $m \geq 1$, so that $\mathrm{lct}_0(f) \leq 1/m \leq 1$. □

Example 1.5. Consider the cusp $x^2 - y^3 = 0$ from before:



$x^2 - y^3 = 0$

To calculate the log canonical threshold, we could just try to determine when the integral

$$\int \frac{1}{|x^2 - y^3|^{2\lambda}}$$

is finite by using an appropriate change of coordinates. The log canonical threshold ends up being

$$\mathrm{lct}_0(x^2 - y^3) = \frac{5}{6},$$

but we'll compute this after introducing another way to compute log canonical thresholds.

1.2 Formula for log canonical thresholds in terms of log resolutions

Let $f \in \mathbf{C}[z_1, \ldots, z_n]$. The idea is that log resolutions give "good" changes of coordinates that allow us to easily calculate log canonical thresholds.

Definition 1.6. $\pi \colon Y \to \mathbf{A}^n$ is a *log resolution of f at* 0 if

- π is a proper birational morphism;
- For every point $p \in \pi^{-1}(0)$, both $f \circ \pi$ and $\mathrm{Jac}_{\mathbf{C}}\, \pi$ are locally monomial, that is, there exists local coordinates $y_1, \ldots, y_n \in \mathcal{O}_{Y,p}$ such that $f \circ \pi = u y_1^{a_1} \cdots y_n^{a_n}$ and $\mathrm{Jac}_{\mathbf{C}}(\pi) = v y_1^{k_1} \cdots y_n^{k_n}$ for locally invertible u, v.

Using the notation from the definition above, we claim the following:

Proposition 1.7. *We can calculate* $\mathrm{lct}_0(f)$ *as*

$$\mathrm{lct}_0(f) = \min_i \frac{k_i + 1}{a_i}.$$

Proof. By properness of π, we have that

$$\int \frac{1}{|f|^{2\lambda}} < \infty \text{ at } 0 \iff \int \frac{|\mathrm{Jac}_{\mathbf{C}}(\pi)|^2}{|f \circ \pi|^{2\lambda}} < \infty \text{ at } p \text{ for all } p \in \pi^{-1}(0).$$

By using local coordinates, and the fact that both $f \circ \pi$ and $\mathrm{Jac}_{\mathbf{C}}\, \pi$ are locally monomial, integrability of the function above is equivalent to the finitude of

$$\int \frac{|y_1^{k_1} \cdots y_n^{k_n}|^2}{|y_1^{a_1} \cdots y_n^{a_n}|^{2\lambda}} = \int \frac{1}{\prod_i |y_i|^{2\lambda a_i - 2k_i}},$$

which is a finite integral at p if and only if $2\lambda a_i - 2k_i - 1 < 1$ for all i, i.e., $\lambda < \frac{k_i+1}{a_i}$, so that

$$\mathrm{lct}_0(f) = \min_{\substack{i \\ \text{all charts}}} \frac{k_i + 1}{a_i}.$$ □

Note that we are cheating a little here: it's not clear the minimum exists if we take the infimum over all charts. We can fix this by thinking about it another way: using the notation of divisors, we can write

$$\mathrm{div}(f \circ \pi) = \sum a_j D_j \quad \mathrm{div}\, \mathrm{Jac}_{\mathbf{C}}(\pi) = \sum k_j D_j,$$

in which case

$$\mathrm{lct}_0(f) = \min_j \frac{k_j + 1}{a_j}.$$

1.3 The relative canonical divisor

Let $\pi\colon Y \to X$ be a projective birational morphism of smooth varieties of dimension n. Then, we can take a differential form on X and pull it back to Y, that is, we have a morphism $\pi^*\Omega_X \to \Omega_Y$; taking the nth wedge power, we get a morphism

$$\pi^*\omega_X \longrightarrow \omega_Y$$

between canonical line bundles. This is a map of line bundles that is nonzero, since looking at the locus where X and Y are isomorphic, we have an isomorphism of line bundles. Tensoring with the dual of the left hand side, we get a morphism

$$\mathcal{O}_Y \longrightarrow \omega_Y \otimes (\pi^*\omega_X)^\vee,$$

i.e., we get a section of the line bundle $\omega_Y \otimes (\pi^*\omega_X)^\vee$. This section gives a divisor which we denote by $K_{Y/X}$, satisfying the following properties:

- $\operatorname{Exc}(\pi) = \operatorname{Supp}(K_{Y/X})$ ("$K_{Y/X}$ measures where the morphism is not étale");
- Choosing K_Y, K_X such that $\pi_*K_Y = K_X$, we have $K_{Y/X} = K_Y - \pi^*K_X$;
- $\operatorname{div}(\operatorname{Jac}_{\mathbf{C}}(\pi)) = K_{Y/X}$ (by looking where they vanish).

The relative canonical divisor satisfies the following properties which are useful for computations:

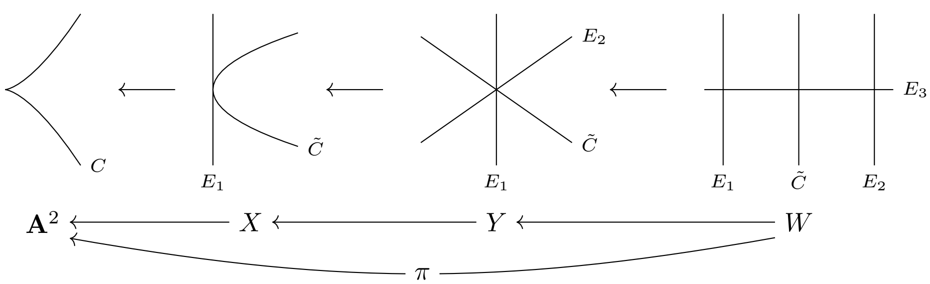
Properties 1.8.

- If $Z \xrightarrow{\phi} Y \xrightarrow{\pi} X$, then $K_{Z/X} = K_{Z/Y} + \phi^*K_{Y/X}$ ("chain rule");
- If $Z \subset X$ is a smooth irreducible subvariety, and if $E \subset B_Z X \to X$ is the blowup of X along Z, then

  $$K_{B_Z X/X} = (\operatorname{codim} Z - 1)E$$

  which you can remember by the fact that blowing up along a codimension 1 subvariety should have a trivial relative canonical divisor.

Example 1.9. We return to the example of the cusp $C = \{x^2 - y^3 = 0\}$. Let $\pi\colon W \to \mathbf{A}^2$ be the log resolution:



Then, using the "chain rule" property

$$\pi^*C = \tilde{C} + 2E_1 + 3E_2 + 6E_3 \quad \text{and} \quad K_{W/\mathbf{A}^2} = E_1 + 2E_2 + 4E_3.$$

We therefore have

$$\operatorname{lct}_0(x^2 - y^3) = \min_i \frac{k_i + 1}{a_i} = \min\left\{\frac{0+1}{1}, \frac{1+1}{2}, \frac{2+1}{3}, \frac{4+1}{6}\right\} = \frac{5}{6}.$$

2 Hypersurface thresholds

We first make the following definition:

Definition 2.1. The *hypersurface thresholds* are defined to be the possible log canonical thresholds for hypersurfaces, that is:

$$\mathcal{HT}_n = \{\operatorname{lct}_0(f) \mid f \in \mathbf{C}[z_1, \ldots, z_n]\}.$$

If $n = 0$, then we define $\mathcal{HT}_0 \coloneqq \{0\}$, following the convention that $\operatorname{lct}_0(0) = 0$.

Example 2.2. By our calculation in Example 1.3, we have

$$\mathcal{HT}_1 = \{0, 1, 1/2, 1/3, 1/4, \ldots\}$$

since $\operatorname{lct}_0(f) = \operatorname{lct}_0(z^a u) = 1/a$ in local coordinates.

Now we observe that

- $\mathcal{HT}_1$ satisfies ACC, that is, there are no infinite increasing sequences in this set.
- The accumulation points of $\mathcal{HT}_1$ is $\mathcal{HT}_0$.

The following Theorem says that these observations hold in general.

Theorem 2.3 (de Fernex–Mustaţă, Kollár, de Fernex–Mustaţă–Ein, Hacon–McKernan–Xu)**.** *$\mathcal{HT}_n$ satisfies ACC, and the accumulation points of $\mathcal{HT}_n$ is $\mathcal{HT}_{n-1} \setminus \{1\}$.*

The idea behind the proof is to consider all polynomials, and somehow think about how a sequence of polynomials has a limit point, called the *generic limit*. To make this precise, we first need some more facts about log canonical thresholds.

2.1 More properties of log canonical thresholds

Lecture 2

We first restate the definitions for log resolutions and log canonical thresholds using the language of divisors, including the relative canonical divisor.

Definition 2.4. $\pi\colon Y \to \mathbf{A}^n$ is a *log resolution of f at* 0 if

1. π is a proper birational morphism;
2. Y is regular;
3. $\operatorname{Exc}(\pi)$ is a divisor;
4. $\operatorname{Exc}(\pi) + \operatorname{div}(f \circ \pi)$ has simple normal crossings.

Remark 2.5. Log resolutions exist in the settings we will be interested in:

1. These exist by Hironaka if k is algebraically closed of characteristic 0
2. We may replace $\mathbf{A}^n$ with $\hat{\mathbf{A}}^n$ (i.e., $k[x_1, \ldots, x_n]$ with $k[[x_1, \ldots, x_n]]$) by Temkin.

We glossed over how to discuss lct_0 in the new language, so we do this now.

Definition 2.6. If $f \in k[x_1, \ldots, x_n]$, and $\pi\colon Y \to \mathbf{A}^n$ is a log resolution, and

$$K_{Y/\mathbf{A}^n} = \sum k_i D_i \quad \text{and} \quad \operatorname{div}(f \circ \pi) = \sum a_i D_i$$

then

$$\operatorname{lct}_0(f) = \min_{i \mid 0 \in \pi(D_i)} \frac{k_i + 1}{a_i} \quad \text{and} \quad \operatorname{lct}(f) = \min_i \frac{k_i + 1}{a_i}.$$

We will be using the same definition for hypersurface singularities in $\hat{\mathbf{A}}^n$.

Proposition 2.7. *If $f \in k[[x_1, \ldots, x_n]]$ and $f(0) = 0$, then*

$$\operatorname{lct}_0(f) \le \frac{n}{\operatorname{ord}_0(f)}.$$

Proof. Choose a log resolution

$$\begin{array}{ccc} Y & \xrightarrow{\ \pi\ } & \hat{\mathbf{A}}^n \\ & \searrow \quad \nearrow & \\ & E \subset B_0\hat{\mathbf{A}}^n & \end{array}$$

Then, $K_{Y/\hat{\mathbf{A}}^n} = (n-1)E_0 + \cdots$, and $\operatorname{div}(f \circ \pi) = \operatorname{ord}_0(f)E_0 + \cdots$. Thus,

$$\operatorname{lct}_0(f) \le \frac{(n-1)+1}{\operatorname{ord}_0(f)}.$$ □

We will use the following result (which we will not prove):

Proposition 2.8 (Demailly–Kollár)**.** *Let f, g be power series such that $f(0) = g(0) = 0$. Then*

$$\operatorname{lct}_0(f+g) \leq \operatorname{lct}_0(f) + \operatorname{lct}_0(g).$$

Sidenote: lct_0 gives a norm on power series $|f-g| = \operatorname{lct}_0(f-g)$.

Corollary 2.9. *Suppose $f \in k[[x_1, \ldots, x_n]]$, $f(0) = 0$. Set $t_m(f)$ to be the mth truncation of f, which is the polynomial consisting of the terms in f of order $\leq m$. Then,*

$$|\operatorname{lct}_0(f) - \operatorname{lct}_0(t_m(f))| \leq \frac{n}{m+1}.$$

Proof. Apply Propositions 2.7 and 2.8 to $t_m(f)$ and $f - t_m(f)$. Then,

$$\operatorname{lct}_0(f) \leq \operatorname{lct}_0(t_m(f)) + \operatorname{lct}_0(f - t_m(f)),$$

and rearranging, we obtain

$$\operatorname{lct}_0(f) - \operatorname{lct}_0(t_m(f)) \leq \operatorname{lct}_0(f - t_m(f)) \leq \frac{n}{m+1}.$$

You can get the other inequality by breaking up f differently. □

Proposition 2.10. *Suppose $f(\vec{x})$, $g(\vec{y})$ are power series in disjoint variables. Then,*

$$\operatorname{lct}_0(f(\vec{x}) + g(\vec{y})) = \min\{1, \operatorname{lct}_0(f) + \operatorname{lct}_0(g)\}.$$

The idea is to take log resolutions for f and g, and then take their product, and then compare the monomials that show up.

Examples 2.11. We can use Proposition 2.10 to easily compute some log canonical thresholds:

- $\operatorname{lct}_0(x^2 - y^3) = \min\{1, \frac{1}{2} + \frac{1}{3}\} = 5/6$.
- Choose $f(\vec{x})$ and $\operatorname{lct}_0(f(\vec{x})) < 1$. Then, if $m \gg 0$,

  $$\operatorname{lct}_0(f(\vec{x}) + y^m) = \operatorname{lct}_0(f) + \frac{1}{m}.$$

  Thus, $\operatorname{lct}_0(f(\vec{x}) + y^m) \to \operatorname{lct}_0(f)$ from above (recall that ACC says that we cannot have this limit from below).

2.2 Accumulation points of hypersurface thresholds

We now return to hypersurface thresholds, which we recall are defined as

$$\mathcal{HT}_n = \{\operatorname{lct}_0(f) \mid f \in k[z_1, \ldots, z_n],\ f(0) = 0\}.$$

We will first show the second statement in Theorem 2.3:

Theorem 2.3∗. *The accumulation points of $\mathcal{HT}_n$ is $\mathcal{HT}_{n-1} \setminus \{1\}$.*

One inclusion is easy:

Proof of "⊇". If $f \in k[x_1, \ldots, x_{n-1}]$ has $\operatorname{lct}_0(f)$, then $f + x_n^m$ will have $\operatorname{lct}_0(f + x_n^m)$ converging to $\operatorname{lct}_0(f)$ by Example 2.11. □

For the other direction, let $f_i \in k[x_1, \ldots, x_n]$, $f(0) = 0$, and $\operatorname{lct}_0(f_i) \to c$. We want to find $F \in K[[x_1, \ldots, x_n]]$ such that $f_i \to F$, $\operatorname{lct}_0(f_i) \to \operatorname{lct}(F)$, where $k \subset K$ has countably infinite transcendence degree. We first mention that the naïve choice for such a limit does not work:

Example 2.12. Consider $f_m = x^2 + \frac{x}{m}$. Then, you would want to say that $f_m \to x^2$ as $m \to \infty$, but this is a bad notion since $\mathrm{lct}(f_m) = 1$, while $\mathrm{lct}(x^2) = 1/2$. Instead, we will consider $F = x^2 + ax \in k(a)[x]$ as a limit for the sequence f_m, for some extra transcendental element a.

We will instead construct what is called the generic limit, which is based on the following observation:

Reminder 2.13. If $Z \subset \mathbf{A}^n_k$ is a closed set, then Z gives rise to an n-tuple in $K(Z)$, since there is a map $\operatorname{Spec} K(Z) \to Z \to \mathbf{A}^n_k$, giving a map $k[x_1, \dots, x_n] \to K(Z)$. Moreover, suppose we have a diagram

$$\begin{array}{ccc} Z_2 & \subseteq & \mathbf{A}^{n+d} \\ {\scriptstyle\text{dominant}}\ \downarrow & & \downarrow \\ Z_1 & \subseteq & \mathbf{A}^n \end{array}$$

This gives an injection of fraction fields $K(Z_1) \hookrightarrow K(Z_2)$, and the process above commutes with such an injection.

We will now construct a "generic limit" F for the sequence f_i.

2.2.1 Generic limits: Easy case

Consider a collection $\{f_i\}_{i\in\mathbf{N}}$ of polynomials with $f_i(0) = 0$, such that all non-zero coefficients are in degree $\leq d$. You can then consider the following finite dimensional vector space parametrizing the coefficients of these polynomials:

$$P_d := k[[x_1, \dots, x_n]]/(x_1, \dots, x_n)^{d+1}.$$

Since this is a finite dimensional vector space, we can also view it as an affine space, in which case each polynomial f_i defines a point $[f_i] \in P_d$. Now we choose $I_d \subsetneq \mathbf{N}$ such that

1. $Z_d = \overline{\{[f_i] \mid i \in I_d\}}$ is irreducible;
2. For any closed subset $Y \subsetneq Z_d$, there are finitely many $[f_i]$ with $i \in I_d$ inside Y.

We can do this by using the noetherian property on P_d.

We now use the following

Fact 2.14 (Log canonical thresholds in families)**.** Consider a set $Z \subset P_d$. Then, there exists an open subset $U \subset Z$ such that $U \ni [f] \mapsto \mathrm{lct}_0(f)$ is constant.

Using this fact and property (2) above, for all but finitely many $i \in I_d$, we have that $\mathrm{lct}_0(f_i)$ is constant.

Definition 2.15. We say that $Z_d \subseteq P_d$ is the *generic limit*. Also, by Reminder 2.13, Z_d gives an element $F \in K(Z_d)[x_1, \dots, x_n]$, which we also call the generic limit.

Note that $\mathrm{lct}_0(F) = \mathrm{lct}_0(f_i)$ for infinitely many $i \in I_d$ by construction. Also, note that since the Z_d are not unique, we are really finding one accumulation point of the sequence, but it will not be a unique limit since F depends on Z_d.

2.2.2 Generic limits: General case

Consider a set $\{f_i\}$ with $f_i(0) = 0$ as before. We can truncate all of these to level d to obtain a new set of polynomials $\{t_d(f_i)\}$. Performing this process for each level d, we get the following tower of P_i's:

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ Z_3 & \subset & P_3 \\ \downarrow & & \downarrow \\ Z_2 & \subset & P_2 \\ \downarrow & & \downarrow \\ Z_1 & \subset & P_1 \end{array}$$

The content of the Lemma below is that we can choose the sets Z_d constructed in the easy case compatibly with this tower of P_i's, giving the diagram above.

Lemma 2.16. *There exist sets $I_1 \supset I_2 \supset I_3 \supset \cdots$ of indices such that*

1. $Z_d = \overline{\{t_d(f_i) \mid i \in I_d\}}$ *is irreducible;*
2. *If $Y \subsetneq Z_d$ is a closed set, it contains only finitely many $[t_d(f_i)]$ with $i \in I_d$.*
3. $Z_{d+1} \dashrightarrow Z_d$ *is dominant.*

Note $K(Z_1) \hookrightarrow K(Z_2) \hookrightarrow \cdots$ and set $K = \bigcup K(Z_d)$. Then, Z_d defines a polynomial $F_d \in K(Z_d)[x_1, \ldots, x_n]$, and $t_d F_{d+1} = F_d$ because the diagram above commutes. We then set $F = \lim F_d$, and claim

Claim 2.17. $\operatorname{lct}_0(F) = \lim_i \operatorname{lct}_0(f_i)$.

As in the easy case, it is actually true that for each d,

$$\operatorname{lct}_0(t_d(F)) = \operatorname{lct}_0(F_d) = \operatorname{lct}_0(t_d(f_i))$$

for infinitely many $i \in I_d$.

2.3 Conclusion of proof of Theorem 2.3∗

Lecture 3

Recall we wanted to show that the limit points of $\mathcal{HT}_n$ is $\mathcal{HT}_{n-1} \setminus \{1\}$. The generic limit construction started with a set $\{f_i\}_{i\in\mathbf{N}}$ such that $f_i \in k[x_1, \ldots, x_n]$, and assuming the sequence $\{\operatorname{lct}_0(f_i)\}$ was non-constant, we constructed a power series $F \in K[[x_1, \ldots, x_n]]$ over a field extension K of k. Recall that the construction actually produced power series $t_d(F)$ that arose as the generic points of closed sets $\overline{\{t_d(f_i) \mid i \in I_d\}}$ for $\mathbf{N} \supseteq I_1 \supset I_2 \supset I_3 \supset \cdots$.

Example 2.18. Let $f_i = x^2 + y^i$; then $\operatorname{lct}_0(f_i) = \frac{1}{2} + \frac{1}{i}$ for $i \geq 2$. Then, we can set $F = x^2$ and $\operatorname{lct}_0(f_i) \to \operatorname{lct}(F)$.

The key property of the construction from last time is that $\lim \operatorname{lct}_0(f_i) = \operatorname{lct}_0(F)$, as is illustrated in this example.

To finish the proof, we recall the following definition:

Definition 2.19. We say $D_{i'}$ *computes* $\operatorname{lct}_0(f)$ if $\operatorname{lct}_0(f) = \frac{k_{i'}+1}{a_{i'}}$. We say that the *center* of $D_{i'}$ on X is the image $\pi(D_{i'})$ under π.

So let f be a holomorphic function such that $f(0) = 0$ and $\{f = 0\}$ has an isolated singularity at the origin. If $Y \to \hat{\mathbf{A}}^n$ is a log resolution of f, then it also is for the truncations $t_d(f)$ for $d \gg 0$. Thus, $\operatorname{lct}_0(t_d(f)) = \operatorname{lct}_0(f)$ for $d \gg 0$: you can show that the coefficients showing up in $K_{Y/\hat{\mathbf{A}}^n}$ and $\operatorname{div}(f \circ \pi)$ are constant for $d \gg 0$. The following Theorem encodes the information of centers into this remark:

Theorem 2.20 (Kollár, de Fernex–Mustata–Ein)**.** *If $f \in K[[x_1, \ldots, x_n]]$ and $f(0) = 0$, and there exists a log resolution $Y \to \hat{\mathbf{A}}^n$ with an exceptional divisor E such that*

- *E computes $\operatorname{lct}_0(f)$, and*
- *$\pi(E) = 0$,*

then $\operatorname{lct}_0(f) = \operatorname{lct}_0(t_d(f))$ for $d \gg 0$.

We can now prove the rest of Theorem 2.3∗.

Proof of "$\subseteq$". Consider our set of polynomials $\{f_i\}$ with generic limit F. We choose a log resolution of F:

$$E \subset Y \xrightarrow{\pi} \hat{\mathbf{A}}^n,$$

where E computes $\operatorname{lct}_0(F)$. There are two cases:

Case 1. $\pi(E) = \{0\}$.

In this case, $\operatorname{lct}_0(F) = \operatorname{lct}_0(t_d(F))$ for $d \gg 0$, which in turn is equal to $\operatorname{lct}_0(t_d(f_i))$ for infinitely many $i \in I_d$ (the idea is that you can extend the exceptional divisor computing $\operatorname{lct}_0(t_d(F))$ to ones computing $\operatorname{lct}_0(t_d(f_i))$). Then, this equals $\operatorname{lct}_0(f_i)$ by using Theorem 2.20.

Case 2. $\pi(E) \supsetneq \{0\}$.

First, localize at the generic point of $\pi(E)$, and take the completion there, to get a complete regular local ring of dimension $n - \dim(\pi(E))$. We denote the image of F by F^*, in which case $\mathrm{lct}_0(F) = \mathrm{lct}_0(F^*) = \mathrm{lct}_0(t_d F^*) \in \mathcal{HT}_{n-\dim \pi(E)}$ (you need to show that localization and completion don't change the numerics of the exceptional divisor computing the log canonical threshold), and so we are done by induction. $\square$

3 A more general setting

We now would like to explain a slightly different perspective for thinking about log canonical thresholds.

Previously, we discussed pairs $(\mathbf{A}^n, \{f = 0\})$, and we looked at singularities of that polynomial. We can replace this with $(X, \mathfrak{a})$, where X is a smooth variety, and $\mathfrak{a}$ is an ideal sheaf.

Definition 3.1. A *divisor over* X is the data $E \subset Y \xrightarrow{\pi} X$, where

- π is proper birational;
- Y is normal;
- E is a prime divisor.

Note that $\mathcal{O}_{Y,E}$ is a DVR, and so we get a valuation ord_E, where $\mathrm{ord}_E(\mathfrak{a}) = e$ if e is the unique number such that $\mathfrak{a} \cdot \mathcal{O}_{Y,E} = (t^e)$ for a uniformizing parameter t for $\mathcal{O}_{Y,E}$.

Definition 3.2. $A_X(\mathrm{ord}_E)$ is the *log dicrepancy* $1 + \mathrm{ord}_E(K_{Y/X})$. We identify two divisors E, E' if $\mathrm{ord}_E = \mathrm{ord}_{E'}$. (Exercise: if E, E' are divisors over X that are identified, then $A_X(\mathrm{ord}_E) = A_X(\mathrm{ord}_{E'})$.)

Then,

$$\mathrm{lct}(\mathfrak{a}) = \min_{E \text{ over } X} \frac{A_X(\mathrm{ord}_E)}{\mathrm{ord}_E(\mathfrak{a})}.$$

Note 3.3 (Zariski).

$$\{\text{Divisors over } X\} \leftrightsquigarrow \left\{\begin{array}{c} \text{DVR's } (R, \mathfrak{m}_R) \text{ of } K(X) \text{ such that} \\ \mathrm{tdeg}(R/\mathfrak{m}_R, k) = \dim X - 1 \\ \text{together with a map} \\ \operatorname{Spec} R \to X \end{array}\right\}$$

The left direction you can obtain by blowing up the image of the map $\operatorname{Spec} R \to X$:

$$\begin{array}{ccccc} & & \operatorname{Spec} R & \longrightarrow & X \\ & & \downarrow & \searrow & \nearrow \\ \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 \end{array}$$

and part of the claim is that this process terminates.

We can also define this using *all* $\mathbf{R}$-valued valuations, instead of restricting to just divisorial ones.