Learning Seminar on Deligne’s Weil II Theorem

Organized by Bhargav Bhatt

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Contents

1 May 3—Overview (Bhargav Bhatt) ......................................................... 2
2 May 11—Weil Sheaves (Tyler Foster) ................................................... 7
3 May 18—Weights I (Brandon Carter) ................................................... 11
4 May 23—Weights II (Takumi Murayama) ............................................ 14
5 May 25—Monodromy (Emanuel Reinecke) .......................................... 19
6 May 30—Real Sheaves (Matt Stevenson) ............................................ 26
7 June 1—Fourier Transforms (Charlotte Chan) ...................................... 29
8 June 6—Weil Conjectures I (Bhargav Bhatt) ........................................ 37
9 June 8—Weil Conjectures II & Hard Lefschetz (Bhargav Bhatt) .......... 40

*Notes were taken by Takumi Murayama, who is responsible for any and all errors. David Speyer, Matt Stevenson, and Emanuel Reinecke contributed notes for May 25. Charlotte Chan contributed notes for June 1 and June 6.
May 3—Overview (Bhargav Bhatt)

The goal of the seminar is to prove the Riemann hypothesis part of the Weil conjectures. Today, we will formulate the statements and talk about how they can be interpreted using étale cohomology.

1.1 Weil conjectures

We will use notation from Deligne and our reference [KW01].

**Notation 1.1.** $X_0$ will denote a variety over $\mathbb{F}_q$, and $X := X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ will denote the corresponding variety over the algebraic closure of $\mathbb{F}_q$.

The idea is we want to compute how many $\mathbb{F}_q$-points there are on $X_0$. We do this by putting everything in a formal power series:

**Definition 1.2.** The zeta function for $X_0$ is defined as the formal power series

$$Z(X_0, t) = \exp\left(\sum_{r=1}^{\infty} \frac{\#X_0(\mathbb{F}_q^r)}{r} t^r\right) \in \mathbb{Q}[t].$$

**Example 1.3.** Let $X_0 = \text{Spec}(\mathbb{F}_q)$. We get

$$Z(X_0, t) = \exp\left(\sum_{r=1}^{\infty} \frac{1}{r} t^r\right) = \frac{1}{1-t}.$$

**Example 1.4.** Let $X_0 = \mathbb{P}^1$. Then, $\#X_0(\mathbb{F}_q^r) = 1 + q^r$ by using the decomposition $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, and so

$$Z(X_0, t) = \exp\left(\sum_{r=1}^{\infty} \frac{(1 + q^r)}{r} t^r\right) = \frac{1}{(1-t)(1-qt)}.$$

These are both rational functions: this is the first part of the Weil conjectures. Motivated by slightly more complicated examples (Fermat hypersurfaces), Weil formulated the following conjectures:

**Conjectures 1.5** (Weil). Let $X_0$ be a smooth projective variety over $\mathbb{F}_q$, which is geometrically connected of dimension $n$, and let $Z(t) := Z(X_0, t)$.

1. **Rationality:** $Z(t)$ is a rational function.
2. **Functional equation:** $Z(1/q^n t)$ and $Z(t)$ are related up to a “fudge factor”:

$$Z\left(\frac{1}{q^n t}\right) = \pm q^{nE/2} t^E \cdot Z(t),$$

where $E$ is a number coming from the geometry of $X$ (or $X_0$): $E = \Delta \cdot \Delta$, where $\Delta \subset X_0 \times X_0$ is the diagonal. [Alternatively, $E = c_{\text{top}}(T_{X_0})$ is the top Chern class of the tangent bundle, or $E = \chi_{\text{top}}(X)$ is the topological Euler characteristic (this uses 4).]
3. **Riemann hypothesis:** The rational function $Z(t)$ has a special form:

$$Z(t) = \frac{P_2(t)P_3(t) \cdots P_{2n-1}(t)}{P_0(t)P_2(t) \cdots P_{2n}(t)},$$

where each $P_i(t)$ satisfies the following properties:

(a) $P_0(t) = 1 - t \in \mathbb{Z}[t]$;
(b) $P_{2n}(t) = 1 - q^{n} t \in \mathbb{Z}[t]$;
(c) For $1 \leq i \leq 2n - 1$, we have

$$P_i(t) = \prod_j (1 - \alpha_{ij} t) \in \mathbb{Z}[t],$$

where each $\alpha_{ij}$ is an algebraic integer, and $|\alpha_{ij}| = q^{i/2}$, where $|\cdot|$ denotes the complex norm for any embedding of $\mathbb{Z}[\alpha_{ij}]$ in $\mathbb{C}$. 


4. **Betti numbers:** If $X_0$ lifts to some $Y$ in characteristic zero (i.e., its mod $p$ reduction is $X_0$), then $\deg P_i(t) = \beta_i(Y \otimes \mathbb{C})$, the $i$th (topological) Betti number of $Y \otimes \mathbb{C}$.

We will assume 1 and 2 to be known (they are covered in introductory courses on étale cohomology), and focus on 3. We can see the Weil conjectures hold by inspection for the two examples above. In particular, for 4, you can see how the singular cohomology of $\mathbb{P}^1$ shows up in the zeta function for $\mathbb{P}^1$.

For particular examples, you can often see how to write down the lifting for 4 (Weil probably did this for hypersurfaces).

### 1.2 Review of étale cohomology

We now recall Grothendieck’s formalism for tackling these conjectures.

#### 1.2.1 Fundamental groups

To begin, to define an algebraic analogue of singular cohomology, we look at fundamental groups (note this is not how it happened historically).

Let $X/k$ be a geometrically connected variety, and fix a base point $x: \text{Spec}(k) \to X$.

1. There exists a canonical profinite group $\pi_1(X, x)$ (independent in $x$, up to conjugation) such that it satisfies the following universal property:

    $$(\{\text{finite } \pi_1(X, x)\text{-sets}\} + \text{the forgetful functor}) \cong (\{\text{finite étale covers } Y \to X\} + \text{fibre over } x)$$

The construction of such a group is in SGA1.

**Example 1.6.**

(a) $X = \text{Spec } k$. Then, $\pi_1(X, x) \cong G_k = \text{Gal}(\overline{k}/k)$

(b) $X = \mathbb{P}^1$. By Riemann–Hurwitz, there are no non-trivial finite covers of $\mathbb{P}^1_k$, and so $\pi_1(X, x) \cong G_k$.

In this way, you can see that the fundamental group generalizes the Galois group to schemes.

2. If $X/k$ is geometrically connected, with base point $x$, then

$$1 \longrightarrow \pi_1(X \otimes \overline{k}, x) \longrightarrow \pi_1(X, x) \longrightarrow \pi_1(\text{Spec}(k)) \longrightarrow 1$$

$$\text{mod } \pi_1^{\text{geom}}(X) \text{ mod } \pi_1^{\text{arith}}(X) \text{ mod } G_k$$

where the notation in the second line is due to Katz.

#### 1.2.2 Local systems

Now that we have fundamental groups, we can look at their representations as we do in topology to get local systems.

Fix a prime $\ell$ invertible in $k$. We work with $\overline{Q}_\ell$ coefficients. We then get a category $\text{Loc}(X, \overline{Q}_\ell) =: \text{Loc}(X)$.

**Fact 1.7.** Assume $X$ is normal. Then, $\text{Loc}(X) \cong \text{Rep}_{\text{Cont}}^{\overline{Q}_\ell}(\pi_1(X, x))$, where $\overline{Q}_\ell$, defined by $F \mapsto F_{\overline{k}}$, is the $\ell$-adic Tate module.

**Example 1.8** (Tate twist). We ignored this for the most part in MATH731—Perverse Sheaves.

We define $\overline{Q}_\ell(1) := \overline{Q}_\ell \otimes \mathbb{Z}_\ell \mathbb{Z}_\ell(1)$, where

$$\mathbb{Z}_\ell(1) = \lim_{\overset{\longrightarrow}{n}} \mu_{\ell^n} = T_\ell(\overline{k}^*)$$

is the $\ell$-adic Tate module.

**Example 1.9.** Let $k = \mathbb{F}_q$, and $\pi_1(\text{Spec}(k)) = G_{\mathbb{F}_q} \cong \hat{\mathbb{Z}}$. Let $\text{Frob}_{\mathbb{F}_q}$ be the geometric Frobenius (the inverse of raising functions to the $q$th power), which is an element of $G_{\mathbb{F}_q}$. Then, $\text{Frob}_{\mathbb{F}_q} \otimes \overline{Q}_\ell(-1)$ by multiplication by $q$. The reason for this convention comes later when it will turn out to have positive weight.
1.2.3 Constructible sheaves

Local systems end up not being enough, so we expand our world to “constructible $\mathcal{Q}_ℓ$-sheaves”: $\text{Cons}(X) := \text{Cons}(X, \mathcal{Q}_ℓ)$. Note that $\text{Cons}(X) \supset \text{Loc}(X)$.

Facts 1.10.
1. If $F \in \text{Cons}(X)$, there exists a stratification $\{Z_i \to X\}$ such that $F|_{Z_i} \in \text{Loc}(Z_i)$.
2. D\text{é}vissage: if you want to prove things about constructible sheaves, you can reduce to the local system case. If $U \xrightarrow{i} X \xleftarrow{j} Z$ open/closed decomposition, we get a short exact sequence
   $$0 \to j_!(F|_U) \to F \to i_*(F|_Z) \to 0.$$ 

   This will be useful to reduce the Weil conjectures to the case of local systems.
3. We have the following inclusions:

   $$\text{Loc}(X) \subset \text{Cons}(X) \subset \text{Mod}(X, \mathcal{Q}_ℓ)$$

   $$\text{D}^b_{\text{cons}}(X, \mathcal{Q}_ℓ) \subset \text{D}(\text{Mod}(X, \mathcal{Q}_ℓ))$$

   $$\text{D}(X)$$

   If you want to see this stuff done precisely, you should look the pro-étale stuff. You can do this more categorically/intrinsically.

4. The six functors. Let $X/k$ be a variety. Then,
   (a) There are bifunctors $\otimes$ and $\text{RHom}(-, -)$ on $\text{D}(X)$;
   (b) If $f : X \to Y$ is a morphism, there are functors

   $$f_!, f_* : \text{D}(X) \to \text{D}(Y), \quad f^!, f^* : \text{D}(Y) \to \text{D}(X)$$

   such that $(f_!, f^!)$ and $(f^*, f_*)$ form adjoint pairs (the left one in each ordered pair is a left adjoint, and the right one is a right adjoint).

   These work well with base change.

Definition 1.11. Let $k = \overline{k}$, $K \in \text{D}(X)$ (e.g., $K = \mathcal{Q}_ℓ$). Then,

   $$H^i(X, K) := H^i(f_!K), \quad H^i_!(X, K) := H^i(f_*K),$$

   where $f$ is the structure morphism $X \to \text{Spec} k$.

Note 1.12. If $X_0/\mathbb{F}_q$, $K \in \text{D}(X_0)$, then both cohomology groups $H^i(X, K)$ and $H^i_!(X, K)$ get canonical actions of the absolute Galois group $G_{\mathbb{F}_q}$ by acting on the second factor $K$.

Goal 1.13. We want to understand these representations of Galois groups on cohomology.

5. Duality. Let $X/k$ be a variety, where $k$ is a finite field or $k = \overline{k}$ (things might be bad when the field has infinite dimensional cohomology), and let $f : X \to \text{Spec}(k)$. To do duality you need a dualizing object; in this case it will be the complex $\omega_X := f^!\mathcal{Q}_ℓ \in \text{D}(X)$, since the constant sheaf $\mathcal{Q}_ℓ$ is a dualizing object on $\text{Spec}(k)$, and $f^!$ takes dualizing objects to dualizing objects. The complex $\omega_X$ satisfies:
   (a) $D_X = \text{RHom}(-, \omega_X)$ induces $D(X)^{\text{op}} \xrightarrow{\sim} D(X)$.
   (b) If $f : X \to Y$ is a morphism, then
      * $D_Yf_! = f_*D_X$;
      * $D_Xf^! = f^*D_Y$.

6. We record a computation: If $X$ is smooth of dimension $n$, then $\omega_X = \mathcal{Q}_ℓ(n)[2n]$, so the dualizing object is the constant sheaf shifted by the (étale cohomological) dimension $2n$. This is similar to how in coherent cohomology for smooth (or even Gorenstein) varieties, the dualizing complex is shifted by the (coherent cohomological) dimension $n$. 

4
Corollary 1.14 (Poincaré duality). If $X$ is smooth of dimension $n$ over $k = \mathbb{F}$, then
\[ H^i(X, \mathbb{Q}_\ell) = H^{2n-i}_c(X, \mathbb{Q}_\ell(n))^\vee = (H^{2n-i}_c(X, \mathbb{Q}_\ell))^\vee (-n). \]

This looks like the regular Poincaré duality except it has a Tate twist; this also shows up when you do duality with Hodge structures.

Proof of Corollary. Set $K = \mathbb{Q}_\ell$, and $f: X \to \text{Spec}(k)$. Then,
\[ D_{pt}(f_\ast K) = f_! D_X(K) = f_! R\text{Hom}(K, \mathbb{Q}_\ell(n)[2n]) = f_! \mathbb{Q}_\ell(n)[2n] \]
Taking $H^{-i}$ on both sides, and using the definition $H^i(X, K) = H^i(f_\ast K)$ and similarly for compactly supported cohomology, we get the statement desired. 

We should have mentioned at some point that $\text{Cons}(pt)$ is the same as finite-dimensional vector spaces.

Example 1.15. Let $X_0/\mathbb{F}_q$ be a smooth affine curve, and $F \in \text{Loc}(X_0)$. We will later see that this is the key computation: you can reduce to the case of curves, but with interesting coefficients.

Theorem 1.16 (Artin). $H^i(X, F) = 0$ for all $i > 1$, i.e., the only interesting groups are $H^0(X, F)$ and $H^1(X, F)$.

One of these is easy by using that $\pi_1^{\text{geom}}(X) \subset \pi_1^{\text{arith}}(X)$, and the statement about representations of fundamental groups we had before:
\[ H^0(X, F) = F\pi_1^{\text{geom}}(X). \]

$H^1(X, F)$ is a bit more interesting, . . .

The dual statement for compactly supported cohomology is:
\[ H^i_c(X, F) = \begin{cases} 0 & i \leq 0, \ i \geq 3 \\ F\pi_1^{\text{geom}}(X) & i = 2 \\ ?? & i = 1 \end{cases} \]

7. Lefschetz trace formula. This is not how it is discussed in SGA, but this is a good way to think of it nowadays. Let $X_0/\mathbb{F}_q$ be a variety, and consider the “sheaf-function correspondence”:
\[ D(X_0) \overset{\phi}{\to} \text{Fun}(X_0(\mathbb{F}_q), \mathbb{Q}_\ell) \\
K \mapsto ((x: \text{Spec}(\mathbb{F}_q) \to X_0) \mapsto \text{Trace}(\text{Frob}_x \mid x^* K)) \]
where if $K \in D(\mathbb{Q}_\ell\text{-vector spaces})$, and $g: K \to K$, then
\[ \text{Trace}(g \mid K) = \sum (-1)^i \text{Trace}(g \mid H^i(K)), \]
where the signs are there to make Trace work well with short exact sequences.

The Lefschetz trace formula says this definition works well with pushforward and pullback.

Theorem 1.17 (Lefschetz trace formula). Let $f: X_0 \to Y_0$ be a morphism over $\mathbb{F}_q$. Then,
(a) $\phi$ commutes with $f^*$ and extension of scalars to $\mathbb{F}_q^r$; 
(b) $\phi$ commutes with $f_!$:
\[ \begin{array}{ccc}
D(X_0) & \overset{\phi}{\to} & \text{Fun}(X_0(\mathbb{F}_q), \mathbb{Q}_\ell) \\
\downarrow f_! & & \downarrow \text{sum } \phi \text{ along fibres} \\
D(Y_0) & \overset{\phi}{\to} & \text{Fun}(Y_0(\mathbb{F}_q), \mathbb{Q}_\ell) 
\end{array} \]

where $\text{Fun}(-, -)$ denotes functions as sets.
Example 1.18. \( Y_0 = \text{Spec}(\mathbb{F}_q), \ K = \overline{\mathbb{Q}}_\ell \in \text{D}(X_0), \) and the theorem says

\[
\begin{array}{c}
\text{D}(X_0) \xrightarrow{\phi} \text{Fun}(X_0(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) \\
\downarrow f_! \quad \downarrow f_!
\end{array}
\]

\[
\text{D}(Y_0) \xrightarrow{\phi} \overline{\mathbb{Q}}_\ell
\]

commutes, and so

\[
f_! \phi(K) \quad \phantom{\quad \text{sum}} \quad \text{sum}
\]

\[
\sum_{x \in X_0(\mathbb{F}_q)} 1 \quad \text{Trace}(\text{Frob}_{\mathbb{F}_q} \mid \text{R} \Gamma_c(X, \overline{\mathbb{Q}}_\ell))
\]

\[
\# X_0(\mathbb{F}_q) \quad \sum (−1)^i \text{Trace}(\text{Frob}_{\mathbb{F}_q} \mid H^i_c(X, \overline{\mathbb{Q}}_\ell))
\]

In this way, the \textit{one} complex \( H^i_c(X, \overline{\mathbb{Q}}_\ell) \) contains the information about all \( \mathbb{F}_q \)-points:

\[
Z(X_0, t) = \prod_{i=0}^{2 \dim(X)} \left( \det(1 - t \cdot \text{Frob}_{\mathbb{F}_q} | H^i_c(X, \overline{\mathbb{Q}}_\ell)) \right)^{(−1)^{i+1}}.
\] (1)

Exercise 1.19. Prove rationality of \( Z(X_0, t) \) using this formula.

Similarly, the functional equation comes from Poincaré duality, and the Betti numbers come from smooth and proper base change.

All this stuff came before the papers we will talk about in this seminar.

1.3 Deligne’s theorem

The theorem discusses the eigenvalues of the operator we have on the right hand side in (1).

Fix \( \iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C} \). We can talk about \( |\iota(\alpha)| = |\alpha|_\iota \).

Definition 1.20.

1. Let \( V \in \text{Rep}_{\overline{\mathbb{Q}}_\ell}^{\text{cont}}(G_{\mathbb{F}_q}) \). Then, \( V \) is \( \iota \)-pure of weight \( w \) if

\[
\begin{aligned}
&\text{each eigenvalue} \\
of \text{Frob}_{\mathbb{F}_q} \circ V
\end{aligned} = \sqrt{(\# \mathbb{F}_q)^w} = q^{w/2}
\]

\((*)\)

2. If \( X_0/\mathbb{F}_q \) is smooth and geometrically connected, and \( F \in \text{Loc}(X) \), then \( F \) is \( \iota \)-pure of weight \( w \) if for all \( x : \text{Spec}(\mathbb{F}_{q^r}) \to X_0 \), the pullback \( x^* F \) is \( \iota \)-pure of weight \( w \).

3. In the setup of 1 or 2, \( F \) is \( \iota \)-mixed of weight \( \leq w \) (or respectively \( \geq w \)), if have \( \leq \) (respectively \( \geq \)) in \((*)\).

Example 1.21. Let \( X_0 \) be a smooth, geometrically connected variety of dimension \( n \). Let \( F \in \text{Loc}(X) \) be \( \iota \)-pure of weight \( w \). Then, \( H^0(X, F) \supset H^0_c(X, F) \) are both \( \iota \)-pure of weight \( w \).

Proof. Choose \( x : \text{Spec}(\mathbb{F}_{q^r}) \to X_0 \), and use \( H^0(X, F) \subset F_x \). \( F_x \) has the statement for all eigenvalues, so \( H^0(X, F) \) does as well.

Duality gives that \( H^{2n}_c(X, F) = H^0(X, F^\vee)^\vee(−n) \) is \( \iota \)-pure of weight \( w + 2n \).

This is the babiest case of Deligne’s theorem; the theorem also gives information for intermediate cohomology groups.
Theorem 1.22 (Deligne). Let $X_0/F_q$ be a variety (not necessarily projective), and let $F \in \text{Loc}(X_0)$ be $\iota$-pure of weight $w$. Then, $H^i(X, F)$ is $\iota$-mixed of weight $\leq w + i$.

A small (purely geometric) argument (to be discussed later) shows that this is equivalent to the same statement for $X_0$ a smooth geometrically connected affine curve (you can even use $\mathbb{A}^1$!), and $i = 1$.

Corollary 1.23. If $X_0$ is a smooth, projective, geometrically connected variety over $F_q$, then $H^i(X, \mathbb{Q}_\ell)$ is $\iota$-pure of weight $i$.

Proof of Corollary. First, $H^i(X, \mathbb{Q}_\ell) = H^i_c(X, \mathbb{Q}_\ell)$ since $X$ is complete, which is $\iota$-mixed of weight $\leq i$ by the Theorem. Second, by duality, $H^i(X, \mathbb{Q}_\ell) = H^{2n-i}(X, \overline{\mathbb{Q}}_\ell)^\vee(-n)$, and the right hand side is $\iota$-mixed of weight $\geq i$ by the Theorem again.

2 May 11—Weil Sheaves (Tyler Foster)

The first thing we need to do is to talk about different actions of Frobenius; it will be important to keep track of these things when we define Weil sheaves and state the Grothendieck trace formulas.

Fix a ground field $F_q = \kappa$, and fix an algebraic closure $k = F_{q \text{alg}}$. The Frobenius morphism $\sigma_k : k \to k$ is defined by $a \mapsto a^q$. The geometric Frobenius is $F = \sigma^{-1}$. The Galois group is $\text{Gal}(k/F_q) \cong \hat{\mathbb{Z}}$, which contains $\sigma$ as a topological generator. The Weil group $W(k/F_q) = \langle F \rangle \cong \mathbb{Z}$ is the subgroup of $\text{Gal}(k/F_q)$ generated by the geometric Frobenius.

We will consider different Frobenius operators on schemes. The first one is the familiar one: Given a $F_q$-scheme $X$, the map $\sigma$ induces a morphism $\sigma_{X/F_q} : X \to X$ which fixes the underlying topological space $|X|$ of $X$, and acts by $q$th powers $f \mapsto f^q$ on the structure sheaf $\mathcal{O}_X$.

Now consider an $F_q$-scheme $X_0$. We get a second Frobenius action

\[ \begin{align*}
X & \xrightarrow{\sigma_{X/F_q}} X \\
& \xleftarrow{\sigma_k/F_q \times \text{id}_{X_0}} X_0 \\
& \xrightarrow{\pi} \text{Spec } k \xleftarrow{\sigma_k/F_q} \text{Spec } F_q
\end{align*} \]

We call this morphism $\text{Fr}_X : X \to X$, the Frobenius endomorphism of $X$. Note it is the unique map such that $\sigma_k/F_q \times \text{id}_{X_0} \circ \text{Fr}_X = \sigma_{X/F_q}$ by using the larger cartesian parallelogram.

Example 2.1. Let $X_0 = \mathbb{A}_F^1$. We are interested in the factorization

\[ \begin{align*}
\mathbb{A}_k^1 & \xrightarrow{\sigma_{\mathbb{A}_k^1/F_q}} \mathbb{A}_k^1 \\
\mathbb{A}_k^1 & \xrightarrow{\text{Fr}_{\mathbb{A}_k^1}} \mathbb{A}_k^1 \\
\sigma_{\mathbb{A}_k^1/F_q} & \xrightarrow{\sigma_k \times \text{id}} \mathbb{A}_k^1
\end{align*} \]

which is dual to

\[ \begin{align*}
k[t] & \xrightarrow{f^q} k[t] \\
\mathbb{A}_k^1 & \xleftarrow{\text{coeff } a} \mathbb{A}_k^1
\end{align*} \]
Now draw the same bottom right square as in (2), but with $F$ replacing $\sigma_k/F_q$:

$$
\begin{array}{c}
\xymatrix{ 
X & X_0 \\
\Spec k & \Spec k \\
\downarrow & \downarrow \\
F \times \id_{X_0} & F \\
F & F \\
\Spec k & \Spec F_q \\
\end{array}
$$

This gives the Frobenius automorphism of $X/F_X : X \to X$ where $F_X = F_x \cdot \id_{X_0}$.

We will discuss Grothendieck’s trace formula in terms of these Frobenius morphisms; Weil sheaves will provide a language to extend it.

Fix $X_0$ a scheme over $F_q$, and fix $G_0$ an (étales) $\mathbb{Q}_\ell$-sheaf. Also fix $x \in |X_0|$ a closed point with residue field $k(x)$ and set $d(x) = [k(x) : F_q]$, and a geometric point

$$
\begin{array}{c}
\xymatrix{ 
\Spec k \ar[r]^\pi & X \\
\Spec k(x) \ar[u] & X_0 \\
} 
\end{array}
$$

Now form the fibre $G_0(x)$ which comes with an action of geometric Frobenius $F : G_0(x) \to G_0(F(x))$, with $d(x)$th power $F_x : G_0(x) \to G_0(x)$.

Now recall that an (étale) $\mathbb{Q}_\ell$-sheaf is represented by an étale $E$-sheaf, where $E$ is some finite field extension of $\mathbb{Q}_\ell$:

$$
\mathbb{Q}_\ell \subset E \subset \overline{\mathbb{Q}}_\ell
$$

and we have an inverse system of finite étale $E$-sheaves $(G_{0i})_{i=1}^\infty$, which represent $G_0$ as a cokernel

$$
\begin{array}{c}
\xymatrix{ 
R_i & G_{0i} \\
\ar[ur] & \ar[ur] \\
X_0 \ar[ur] & \ar[ur] \\
\ar[ur] \ar[ur] & \ar[ur] \\
X \ar[ur] & \ar[ur] \\
\ar[ur] \ar[ur] & \ar[ur] \\
X \ar[ur] & \ar[ur] \\
\ar[ur] \ar[ur] & \ar[ur] \\
X \ar[ur] & \ar[ur] \\
\ar[ur] \ar[ur] & \ar[ur] \\
X \ar[ur] & \ar[ur] \\
\ar[ur] \ar[ur] & \ar[ur] \\
} 
\end{array}
$$

We then have the following commutative diagram

$$
\begin{array}{c}
\xymatrix{ 
G_{0i} \ar[r]^{F \times G_{0i}} & G_{0i} \\
X \ar[r]^F & X \\
\Spec k \ar[u] \ar[r]^\pi & X \\
\Spec k \ar[u] \ar[r]^\pi & X \\
\Spec k(x) \ar[u] \ar[r]^\pi & X_0 \\
\Spec k(x) \ar[u] \ar[r]^\pi & X_0 \\
} 
\end{array}
$$

giving rise to a morphism $\mathcal{G} \to F_X^* \mathcal{G}$.

We now claim that there is an isomorphism $F_X^* \mathcal{G} \cong \mathcal{G}$. First, we have two projection morphisms

$$
\begin{array}{c}
\xymatrix{ 
X \times_{X_0} X \ar[r]^{\text{pr}_1} & X \\
X \ar[r]^{\text{pr}_2} & X_0 \\
\Spec k \ar[u] & \Spec k \ar[u] \\
\Spec k \ar[u] & \Spec k \ar[u] \\
\Spec k(x) \ar[u] & \Spec k(x) \ar[u] \\
\Spec k(x) \ar[u] & \Spec k(x) \ar[u] \\
} 
\end{array}
$$

whose compositions to $X_0$ are equal, giving an isomorphism $\text{pr}_1^* \mathcal{G} \cong \text{pr}_2^* \mathcal{G}$. Now note that

$$
X \times_{X_0} X \cong (k \otimes_{F_q} k) \otimes_{F_q} X_0 \cong \text{Gal}(k, F_q) \times X =: \coprod_{\text{Gal}(k/F_q)} X.
$$
Replacing $X \times_{X_0} X$ with $\text{Gal}(k, \mathbb{F}_q) \times X$ in the diagram above, we get the new diagram

$$\text{Gal}(k, \mathbb{F}_q) \times X \xrightarrow{\alpha \cdot \text{pr}_2} X \xrightarrow{} X_0$$

where $\alpha$ denotes the action of $\text{Gal}(k/\mathbb{F}_q)$ via $F_X$. The isomorphism $\text{pr}_1^* \mathcal{G} \rightarrow \text{pr}_2^* \mathcal{G}$ from above breaks up into $\# \text{Gal}(k/\mathbb{F}_q)$ copies of an isomorphism $g^* \mathcal{G} \rightarrow \mathcal{G}$ for each $g \in \text{Gal}(k/\mathbb{F}_q)$. In particular, for $g \in \text{Gal}(k/\mathbb{F}_q)$ corresponding to the Frobenius morphism $F_X$, we have an isomorphism $F_X^* \mathcal{G} \sim \mathcal{G}$.

By precomposition with the morphism $\mathcal{G} \rightarrow F_X^* \mathcal{G}$ from before, we obtain an endomorphism $\mathcal{G} \rightarrow F_X^* \mathcal{G} \sim \mathcal{G}$. We will look at this morphism fibrewise.

Now we define the $L$-function for $\mathcal{G}_0$ to be

$$L(X_0, \mathcal{G}_0, t) = \prod_{x \in |X_0|} \det \left(1 - t^{\dim(x)} F_x, \mathcal{G}_0|_x \right)^{-1}$$

By using the Frobenius endomorphism $F_X : X \rightarrow X$, and by unpacking what a $\mathbb{Q}_l$-sheaf is as before, we have an isomorphism $\alpha : \mathcal{G} \sim F_X^* \mathcal{G}$ with inverse $\alpha^{-1} : F_X^* \mathcal{G} \sim \mathcal{G}$. Because $F_X : X \rightarrow X$ is proper, we get an induced map on compactly supported cohomology:

$$H^i_c(X, \mathcal{G}) \xrightarrow{F} H^i_c(X, F_X^* \mathcal{G}) \xrightarrow{\alpha^{-1}} H^i_c(X, \mathcal{G})$$

**Theorem 2.2** (Grothendieck trace formula). The $L$-function can be written as a finite product:

$$L(X_0, \mathcal{G}_0, t) = \prod_{i=0}^{2 \dim X} \det \left(1 - t F, H^i_c(X, \mathcal{G}) \right)^{(-1)^{i+1}}.$$  

The key piece of structure that lets us formulate this trace formula, and even just define the $L$-function itself, is the action of Frobenius on stalks of our sheaf, which exists since we started with a $\mathbb{Q}_l$-sheaf that lives on $X_0$. The definition of a Weil sheaf just introduces this action of Frobenius directly, so that it still makes sense to write down its $L$-function.

**Definition 2.3.** A Weil sheaf on $X_0$ consists of

1. A $\mathbb{Q}_l$-sheaf $\mathcal{G}$ on $X$;
2. An isomorphism $F^* : F_X^* \mathcal{G} \sim \mathcal{G}$.

**Notation 2.4.** We will refer to a Weil sheaf $\mathcal{G}_0$ as “$\mathcal{G}$ on $X_0$”, even though the actual sheaf lives on $X$.

**Definition 2.5.** A Weil sheaf $\mathcal{G}_0$ on $X_0$ is smooth of rank $r$ if $\mathcal{G}$ is smooth of rank $r$ on $X$. Bhargav called these lisse in MATH731—Perverse Sheaves.

**Properties 2.6.** Here are some properties of Weil sheaves:

1. Weil sheaves form an abelian category with étale $\mathbb{Q}_l$-sheaves on $X_0$ as a full subcategory;
2. The category does not depend on $\mathbb{F}_q$, that is, given $\mathbb{F}_{q'} \subset \mathbb{F}_q$, restriction of scalars gives a natural equivalence between Weil sheaves on $X_0/\mathbb{F}_q$ to Weil sheaves on $X_0/\mathbb{F}_{q'}$;
3. Weil sheaves have pullbacks, derived direct images, and direct image with compact support;
4. $F^*$ induces a morphism $F : H^i_c(X, \mathcal{G}) \rightarrow H^i_c(X, \mathcal{G})$;
5. Any Weil sheaf comes with an automorphism $F_X : \mathcal{G}_x \sim \mathcal{G}_x$ for a geometric point $x$ over a closed point $x \in |X_0|$.

**Remark 2.7.** In future sections, “sheaf” means Weil sheaf, particularly in [KW01] Chap. I.

We now want to show that the Grothendieck trace formula also holds for these Weil sheaves. To do so, we need to discuss a Tannakian duality between Weil sheaves and representations of $\pi_1(X_0, \pi)$. 


Assume $X_0$ is geometrically connected, and fix a geometric point

$$\xymatrix{ \Spec k \ar[r]^-{\pi} & X_0 \ar@<1ex>[u]^-{\pi} \\ X }$$

We then have the monodromy exact sequence

$$
\begin{array}{c}
1 \longrightarrow \pi_1(X, \bar{\pi}) \longrightarrow \pi_1(X_0, \bar{\pi}) \longrightarrow \Gal(k/\mathbb{F}_q) \longrightarrow 1 \\
\end{array}
$$

$$
\begin{array}{c}
1 \longrightarrow \pi_1(X, \bar{\pi}) \longrightarrow W(X_0, \bar{\pi}) \longrightarrow W(k/\mathbb{F}_q) \longrightarrow 1 \\
\end{array}
$$

Note that $W(k/\mathbb{F}_q) \cong \mathbb{Z}$ is not given the subspace topology relative to $\Gal(k/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$. The group $W(X_0, \bar{\pi})$ is called the Weil group of $X_0$ attached to the base point $\bar{\pi}$.

To understand the difference between $\mathbb{Q}_\ell$-sheaves and Weil sheaves, we state the Tannakian duality in both contexts. For $\mathbb{Q}_\ell$-sheaves, we have

$$
\begin{array}{c}
\{ \text{étale } \mathbb{Q}_\ell\text{-sheaves} \} \xrightarrow{\text{fib}} \{ \text{continuous representations of } \pi_1(X_0, \bar{\pi}) \text{ on } \mathbb{Q}_\ell\text{-vector spaces} \} \\
\end{array}
$$

where continuous means that there exists a finite extension $\mathbb{Q}_\ell \subset E \subset \overline{\mathbb{Q}}_\ell$ with $E$-linear subspace $W \subset V$ such that $V = \overline{\mathbb{Q}}_\ell \otimes_E W$ and $\pi_1(X_0, \bar{\pi})$ acts continuously on $W$ (this is the same as saying the action is continuous by the topology of $\overline{\mathbb{Q}}_\ell$). This induces an equivalence

$$
\begin{array}{c}
\{ \text{smooth étale } \mathbb{Q}_\ell\text{-sheaves} \} \xrightarrow{\text{fib}} \{ \text{finite dimensional continuous representations of } \pi_1(X, \bar{\pi}) \text{ on } \overline{\mathbb{Q}}_\ell\text{-vector spaces} \} \\
\end{array}
$$

For Weil sheaves, we instead have

$$
\begin{array}{c}
\{ \text{Weil sheaves} \} \xrightarrow{\text{fib}} \{ \text{continuous representations of } W(X_0, \bar{\pi}) \text{ on } \overline{\mathbb{Q}}_\ell\text{-vector spaces} \} \\
\end{array}
$$

and restricting to smooth objects, we have

$$
\begin{array}{c}
\{ \text{smooth Weil sheaves} \} \xrightarrow{\text{fib}} \{ \text{finite dimensional continuous representations of } W(X, \bar{\pi}) \text{ on } \overline{\mathbb{Q}}_\ell\text{-vector spaces} \} \\
\end{array}
$$

Special Case 2.8. Smooth rank 1 Weil sheaves on $\Spec \mathbb{F}_q$ are the same thing as characters

$$
\phi: W(k/\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell
$$

$$
\xymatrix{ \mathbb{Z} \ar[r]^-{\phi} & \phi(\mathbb{F}) = b \\ \mathbb{F} }$$

and conversely, any $b \in \overline{\mathbb{Q}}_\ell$ gives a Weil sheaf $\mathcal{L}_b$ on $\Spec \mathbb{F}_q$. We will also use $\mathcal{L}_b$ to denote the pullback of this sheaf to $X_0$.  

10
Theorem 2.9. Let $X_0$ be a scheme over $\mathbf{F}_q$, and let $\mathcal{G}_0 = (F^* \mathcal{G} \simeq \mathcal{G})$ be a Weil sheaf on $X_0$. Then,
1. If $X_0$ is normal and geometrically connected, and if $\mathcal{G}_0$ is irreducible and smooth of rank $r$, then $\mathcal{G}_0$ is an étale $\mathcal{O}_q$-sheaf on $X_0$ if and only if $\mathcal{G}$ is an étale sheaf.

Corollary 2.10. For any smooth, irreducible sheaf $\mathcal{G}_0$, there exists some $\mathcal{L}_0$ and some $\mathcal{F}_0$ an étale sheaf such that $\mathcal{G}_0 \cong \mathcal{F}_0 \otimes \mathcal{L}_0$.

Corollary 2.11 (Grothendieck trace formula for Weil sheaves). Given a smooth Weil sheaf $\mathcal{G}_0$ on $X_0$, define
\[
L(X_0, \mathcal{G}_0, t) = \prod_{x \in |X_0|} \det \left(1 - t^{d(x)} F_x, \mathcal{G}_0^x\right)^{-1}.
\]
Then, we can compute the $L$-function as
\[
L(X_0, \mathcal{G}_0, t) = \prod_{i=0}^{2 \dim X} \det \left(1 - t F_i, H^1_c(X, \mathcal{F})\right)^{(-1)^{i+1}}.
\]

Proof of Corollary. In the irreducible case, $\mathcal{G}_0 = \mathcal{G}_0 \otimes \mathcal{L}_0$ and $\mathcal{F}_0 = \mathcal{F}_0 \otimes \mathcal{L}_0 \mathcal{F}_0$. We can rewrite each factor on the right-hand side of (3) as
\[
\det \left(1 - t^{d(x)} b^{d(x)} F_x, \mathcal{F}_0 \mathcal{F}_0\right).
\]
For (1), each factor becomes
\[
\det \left(1 - t F, H^1_c(X, \mathcal{F} \otimes \mathcal{L}_0)\right) = \det \left(1 - t b F, H^1_c(X, \mathcal{F})\right)
\]
since $\mathcal{L}_0$ is the pullback of a Weil sheaf on Spec $\mathbf{F}_q$. Grothendieck trace formula for $L(X_0, \mathcal{F}_0, bt)$ gives the irreducible case.

For the general case, use the filtration in (2) and the multiplicativity of trace on filtrations.

3 May 18—Weights I (Brandon Carter)

Notation 3.1. Unless otherwise stated, $\kappa = \mathbf{F}_q$, $k = \pi$, $X_0$ is an algebraic (that is, finite type) scheme over $\kappa$, $\mathcal{G}_0$ is a (Weil) sheaf on $X_0$, and $\tau: \overline{\mathbf{Q}} \to \mathbf{C}$ is a chosen isomorphism. For a (closed) point $x \in |X_0|$, we also denote $d(x) = [\kappa(x): \kappa]$, and $N(x) = \# \kappa(x) = q^{d(x)}$.

Definition 3.2. Let $\beta \in \mathbf{R}$. Then,
1. For each closed point $x \in |X_0|$, fix $\pi$ a geometric point lying over $x$. Then, we have the Weil group $W(k/\kappa(x))$ acting on the stalk $\mathcal{G}_0^x$. We say that $\mathcal{G}_0$ is $\tau$-pure of weight $\beta$ if for all $x \in |X_0|$, we have that for all eigenvalues $\alpha$ of the geometric Frobenius morphism $F_x: \mathcal{G}_0^x \to \mathcal{G}_0^x$, we have $|\tau(\alpha)|^2 = N(x)^\beta$.
2. We say $\mathcal{G}_0$ is $\tau$-mixed if there is a finite filtration of subsheaves
\[
0 = \mathcal{G}_0^{(0)} \subset \mathcal{G}_0^{(1)} \subset \cdots \subset \mathcal{G}_0^{(r)} = \mathcal{G}_0
\]
such that $\mathcal{G}_0^{(j)} / \mathcal{G}_0^{(j-1)}$ is $\tau$-pure of some weight.
3. $\mathcal{G}_0$ is (pointwise) pure of weight $\beta$ if is $\tau$-pure of weight $\beta$ for all choices $\tau: \overline{\mathbf{Q}} \to \mathbf{C}$.
4. $\mathcal{G}_0$ is mixed if there exists a finite filtration as in (2) with successive quotients being pure.
Note some higher rank vector bundles are neither \( \tau \)-pure nor \( \tau \)-mixed.

**Remark 3.3.** If \( \mathscr{F}_0 \) is \( \tau \)-pure for every \( \tau \), but the weight depends on \( \tau \), then there exists some \( \mathfrak{b} \in \mathbb{Q}_\ell \) such that \( \mathscr{F}_0 \cong \mathcal{F}_0 \otimes \mathcal{L}_\mathfrak{b} \) with \( \mathcal{F}_0 \) being pure.

**Permanence Properties 3.4.**

1. If \( f_0 : X_0 \to Y_0 \) is a morphism over \( \kappa \), and \( \mathscr{F}_0 \) is a sheaf on \( Y_0 \), then
   - \( f_0^*(\mathscr{F}_0) \) is \( \tau \)-pure of weight \( \beta \) if \( \mathscr{F}_0 \) is \( \tau \)-pure of weight \( \beta \).
   - If \( f \) is surjective, we get an if and only if statement.
2. If \( f_0 : X_0 \to Y_0 \) is finite, and \( \mathscr{F}_0 \) is a sheaf on \( X_0 \), then \( f_0^*(\mathscr{F}_0) \) is \( \tau \)-pure of weight \( \beta \) if \( \mathscr{F}_0 \) is.
3. If \( X_0/\kappa, \mathscr{F}_0 \) a sheaf on \( X_0 \), and \( \kappa'/\kappa \) a finite extension, then \( \mathscr{F}_0 \) is pure of weight \( \beta \) if and only if the pullback on \( X_0 \otimes_\kappa \kappa' \) is.

Note that (1) and (2) imply (3).

**Remark 3.5.** Similar statements for pure and \( \tau \)-mixed sheaves hold, except for one exception: for \( \tau \)-mixed sheaves, the forward direction of the second subbullet of (1) only holds for finite maps in general. [The filtration is probably not preserved.]

**Definition 3.6.** For \( X_0 \) and \( \mathscr{F}_0 \) as before, and for fixed \( \tau \), we define

\[
w(\mathscr{F}_0) = \sup_{x \in |X_0|} \sup_{\alpha \text{ eigenvalue}} \frac{\log(|\tau(\alpha)|^2)}{\log(N(x))}.
\]

If \( \mathscr{F}_0 \) is trivial (i.e., the zero sheaf), set \( w(\mathscr{F}_0) = -\infty \).

We can use weights to talk about zeros and poles of the \( L \)-function.

**Lemma 3.7.** Suppose \( \mathscr{F}_0 \) is a sheaf on \( X_0 \), such that \( w(\mathscr{F}_0) \leq \beta \), i.e., for all \( x \in |X_0| \), and \( \alpha \) an eigenvalue of \( F_x : \mathscr{F}_0 \to \mathscr{F}_0 \), we have

\[
|\tau(\alpha)|^2 \leq N(x)^\beta = q^{d(x)\beta}.
\]

Then, the \( L \)-function

\[
\tau L(X_0, \mathscr{F}_0, t) = \prod_{x \in |X_0|} \tau \det \left( 1 - t^{d(x)} F_x, \mathscr{F}_0 \right)^{-1}
\]

converges for all \( |t| < q^{-\beta/2 - \dim(X_0)} \) and has no zeros or poles in this region.

The idea is that since Grothendieck’s trace formula says that the \( L \)-function is already a meromorphic function, we can use the logarithmic derivative to detect zeroes and poles of the \( L \)-function.

**Proof.** Reduce to the case where \( X_0 \) is affine, reduced, and irreducible. Then,

\[
\frac{\tau L'(X_0, \mathscr{F}_0, t)}{L(X_0, \mathscr{F}_0, t)} = \sum_{x \in |X_0|} \sum_{n=1}^\infty d(x) \text{Tr}(F^n_x) t^{d(x)n-1} = \sum_{n=1}^\infty \left( \sum_{x \in |X_0|} \frac{d(x) \text{Tr}(F^n_x)}{d(x)n} \right) t^{n-1},
\]

where the second equality is by changing the order of summation, and also changing the index of summation \( n \) to \( n/d(x) \). Now by assumption on \( w(\mathscr{F}_0) \), we have a trivial bound on the traces that appear in this sum:

\[
\left| \tau \text{Tr}(F^n_x/d(x)) \right| \leq r \cdot q^{n\beta/2} \quad \text{where} \quad r = \max_{x \in |X_0|} \dim_{\mathbb{Q}_\ell} \mathscr{F}_0.
\]

This gives the bound

\[
\frac{\tau L'(X_0, \mathscr{F}_0, t)}{L(X_0, \mathscr{F}_0, t)} \leq \sum_{n=1}^\infty \left( \sum_{x \in |X_0|} \frac{d(x)}{d(x)n} \right) r \cdot q^{n\beta/2} t^{n-1} = \sum_{n=1}^\infty \#X_0(F_q^n) \cdot r q^{n\beta/2} t^{n-1}.
\]

12
Now Noether normalization implies $\#X_0(F_{q^n}) \leq C \cdot q^{n \dim X_0}$ for some constant $C$, so

$$\frac{\tau L'(X_0, \mathcal{G}_0, t)}{L(X_0, \mathcal{G}_0, t)} \leq \sum_{n=1}^{\infty} C \cdot r q^{(n \dim X_0 + \beta/2)} t^{n-1},$$

which is just a geometric series.

**Lemma 3.8.** Let $X_0$ be a smooth irreducible curve over $k$, with $U_0 \xrightarrow{j_0} X_0$ a nonempty open subset. Denote $S_0 = X_0 \setminus U_0$ to be the complement of $U_0$. Let $\mathcal{G}_0$ be a sheaf on $X_0$ such that the restriction $j_0^*\mathcal{G}_0$ is smooth and $H^2_S(X, \mathcal{G}) = 0$, i.e., $\mathcal{G}$ has no sections supported on the complement $S = X \setminus U$. Then, $w(j_0^*(\mathcal{G}_0)) \leq \beta$ implies $w(\mathcal{G}_0) \leq \beta$.

**Sketch of Proof.** We denote $\mathcal{F}_0 = j_0^*(\mathcal{G}_0)$ in the following.

1. Reduce to the case where $X_0$ is affine and geometrically irreducible, and $j_0, j_0^* \mathcal{G}_0 = \mathcal{G}_0$. Then, $\pi_\alpha^*(\mathcal{G}_0) = 0$ (by using $j_0, j_0^* \mathcal{G}_0 = \mathcal{G}_0$ and affinity), and so we have

$$L(X_0, \mathcal{G}_0, t) = L(U_0, \pi_\alpha^*(\mathcal{G}_0), t) \cdot \prod_{s \in |S_0|} \det((1 - t^{d(s)} F_\alpha)_{\mathcal{G}_0})^{-1} = \det((1 - t^{d(s)} F_\alpha)_{\mathcal{G}_0})^{-1} \cdot \det((1 - t F_\alpha)_{\mathcal{G}_0})$$

by the Grothendieck trace formula.

2. We want to look at $H^2_c(U, \mathcal{G}) = H^2_c(U, \mathcal{F})$. By Poincaré duality, this is $H^0(U, (\mathcal{F}^\vee(1))^\vee)$ where $(1)$ is a Tate twist. Taking the Tate twist out, we get

$$H^2_c(U, \mathcal{F}) = H^0(U, (\mathcal{F}^\vee(1))^\vee) = (\mathcal{F}^\vee)^{\pi_1(U, \mathcal{F}})^\vee(-1) = (\mathcal{F})^{\pi_1(U, \mathcal{F}})^\vee(-1),$$

where dualizing changes invariants to coinvariants.

Now using the right-hand side of (5), we see that the poles of the $L$-function are of the form $\frac{1}{\alpha q}$ where $\alpha$ is an eigenvalue of $F_\alpha \subset (\mathcal{F}^\vee)^{\pi_1(U, \mathcal{F}})^\vee$. This lifts to give an eigenvalue $\alpha^{d(x)}$ on $\mathcal{F}^\vee_{\mathcal{G}_\alpha}$ (this goes back to something Tyler said last time: Frobenius induces $d$th powers on the coinvariants). This implies $|\tau(\alpha)| < q^{\beta/2}$ by the previous Lemma, and so $\left|\frac{1}{\tau(\alpha q)}\right| > q^{-\beta/2 - 1}$, i.e., we have no poles inside a bounded disk.

3. We use the “factorization” (5) from before, where by the previous Lemma, $\tau L(U_0, \mathcal{G}_0, t)$ has no zeros or poles for $|t| \leq q^{-\beta/2 - 1}$, and $\tau L(X_0, \mathcal{G}_0, t)$ has no poles for $|t| \leq q^{-\beta/2 - 1}$. This implies the factor

$$\prod_{s \in |S_0|} \tau((1 - t^{d(s)} F_\alpha)_{\mathcal{G}_0})^{-1}$$

has no poles, either. Thus the eigenvalues of $F_\alpha : \mathcal{G}_\alpha \to \mathcal{G}_\alpha$ are bounded: $|\tau(\alpha)| < q^{-\beta/2 - 1}$.

4. Play the same game with $j_0^*((\mathcal{F}^\vee)^{\pi_1(U, \mathcal{F}})^\vee)$ to get a bound $|\tau(\alpha)| < q^{-\beta/2 - 1/k}$ and take $k \to \infty$.

**Lemma 3.9.** If $X_0$ is a normal, irreducible algebraic scheme over $k$, and $\mathcal{G}_0$ is irreducible and smooth, and $j_0 : U_0 \hookrightarrow X_0$ where $U_0$ is a dense open subscheme of $X_0$, then $j_0^*(\mathcal{G}_0)$ is also irreducible.

**Proof.** $\pi_1(U_0, \mathcal{G}) \to \pi_1(X_0, \mathcal{G})$. Recall from representation theory that if $G \to G/H \to \text{GL}(V)$ is such that the second arrow is an irreducible representation, then the compositum is also an irreducible representation.

The point of all this is to prove the following

**Theorem 3.10 (Semicontinuity).** Let $\mathcal{G}_0$ be a smooth sheaf on $X_0$ and let $j_0 : U_0 \hookrightarrow X_0$ be an open dense subscheme. Then,

1. $w(\mathcal{G}_0) = w(j_0^*(\mathcal{G}_0))$.
2. If $j_0^*(\mathcal{G}_0)$ is $\tau$-pure of weight $\beta$, then $\mathcal{G}_0$ is $\tau$-pure of weight $\beta$.
3. Let $X_0$ be irreducible and normal, and let $\mathcal{G}_0$ be irreducible. If $j_0^*(\mathcal{G}_0)$ is $\tau$-mixed, then $\mathcal{G}_0$ is also $\tau$-mixed.
4. Let $X_0$ be connected, let $j_0^*(\mathcal{G}_0)$ be $\tau$-mixed, and let $\mathcal{G}_0$ be $\tau$-pure of weight $\beta$ at a single point $x \in |X_0|$ (that is, the condition for $\tau$-purity holds only for the eigenvalues of Frobenius on the stalk at $x$). Then, $\mathcal{G}_0$ is $\tau$-pure of weight $\beta$. 

13
Proof.  
1. Assume $X_0$ is irreducible, and replace $X$ by the normalization of $X_{\text{red}}$. Then, if $\dim(X_0) = 1$, Lemma 3.8 from before applies, and we are done. If $\dim(X_0) > 1$, then connect any point $s \in |X_0 \setminus U_0|$ to a point in $U_0$ with a curve and apply Lemma 3.8 again (you need smoothness of $G_0$ to say $H^0_S(X, G) = 0$).
2. Apply (1) to $G_0$ and $G_0^\vee$ to get an upper and lower bound.
3. Apply Lemma 3.9: $j^*_0(G_0)$ is irreducible, and so $j^*_0(G_0)$ is pure; then, apply (2). Note that we possibly have to shrink $U_0$ to make sure $j^*_0(G_0)$ has a filtration by smooth subsheaves.
4. Assume $X_0$ is irreducible, normal, and $G_0$ is irreducible (work with each irreducible component and irreducible constituent of $G_0$). This implies $j^*_0(G_0)$ is $\tau$-pure by (3), and so $G_0$ is $\tau$-pure by (2), and the weight obviously is $\beta$.

We end with some definitions that will be useful later.

Definition 3.11. Let $G_0$ be a sheaf over $X_0$. Define
$$w_{\text{gen}}(G_0) = w(j^*_0(G_0))$$
for any open dense subscheme $U_0 \to X_0$ on which $j^*_0(G_0)$ is smooth.

Note that (1) in the previous Theorem implies this definition does not depend on the open set $U_0$ chosen.

Definition 3.12. We say $G_0$ is $\tau$-real if the characteristic polynomial $\tau \det(1 - F_x t, G_0^x) \in R[t]$ for all $x \in |X_0|$.

Remark 3.13. This has the obvious permanence properties.

We end with a result showing we can reduce to the $\tau$-real case.

Lemma 3.14. Let $G_0$ be a smooth sheaf, $\tau$-pure of weight $\beta$. Then, $G_0$ is a direct summand of a $\tau$-real and $\tau$-pure sheaf of weight $\beta$.

Proof. We would want to take $G_0^\vee \oplus G_0$, but this changes the weight; instead, we use $(G_0^\vee \otimes L_{\tau^{-1}(q^\beta)}) \oplus G_0$.

4 May 23—Weights II (Takumi Murayama)

Notation 4.1. As usual, $\kappa = F_q$, $k = \kappa$, $X_0$ is an algebraic (i.e., finite type) scheme over $\kappa$, $G_0$ is a (Weil) sheaf on $X_0$, and $\tau : \overline{\mathbb{Q}} \sim \mathbb{C}$ always denotes some isomorphism. If $x \in |X_0|$, we also denoted $d(x) = [\kappa(x) : \kappa]$ and $N(x) = \# \kappa(x) = q^{d(x)}$.

Recall from last time:

Definition 4.2. A weight of $G_0$ is the quantity
$$\frac{\log(|\tau(\alpha)|^2)}{\log(N(x))}$$
where $\alpha$ is an eigenvalue of the Frobenius morphism $F_x : G_0^\tau \to G_0^\tau$ at a geometric point $\overline{x}$ lying over some $x \in |X_0|$. The maximal weight of $G_0$ is
$$w(G_0) = \sup_{x \in |X_0|} \sup_{\text{eigenvalue}} \frac{\log(|\tau(\alpha)|^2)}{\log(N(x))}$$
if $G_0 \neq 0$, and $-\infty$ otherwise.

Recall from last time (Lemma 3.7) that the maximal weight governs where the $L$-function converges:
Lemma 4.3. The $L$-function

$$\tau L(X_0, G_0, t) = \prod_{x \in |X_0|} \tau \det \left( 1 - t^d(x) F_x, \mathcal{G}_x \right)^{-1}$$

converges on the ball $|t| < q^{-w(\mathcal{G}_0)/2 - \dim(X_0)}$, and has no zeros or poles in this ball.

The goal of today’s talk is to give an alternate description for what the maximal weight $w(\mathcal{G}_0)$ is, at least in the case when $\mathcal{G}_0$ is a $\tau$-mixed sheaf on a smooth curve. It turns out that $w(\mathcal{G}_0)$ determines the radius of convergence of a certain power series we introduce later.

Notation 4.4. We introduce some new notation: for every $n \in \mathbb{Z}_{>0}$, we let $\kappa_n = F_n^n$ denote the unique degree $n$ extension of $\kappa$ in $k$. $F_n \in \text{Gal}(k/\kappa_n)$ denotes the geometric Frobenius over $\kappa_n$. In this case, we can describe the set of $\kappa_n$ valued points as

$$X_0(\kappa_n) = \text{Hom}_{\text{Spec } \kappa}(\text{Spec } \kappa_n, X_0) = X_0(k)^{F_n}$$

where $X_0(k) = \text{Hom}_{\text{Spec } \kappa}(\text{Spec } k, X_0)$ are the $k$-valued (geometric) points of $X_0$.

Definition 4.5. The key definition for today is the following function:

$$f_{\mathcal{G}_0} = f_{\mathcal{G}_0}^n : \begin{cases} X_0(\kappa_n) \rightarrow \mathbb{C} \\ \pi \mapsto \tau \text{Tr}(F_{x}^{n/d(x)}, \mathcal{G}_x) = \tau \text{Tr}(F_n, \mathcal{G}_x) \end{cases}$$

where $x \in |X_0|$ is a closed point, and $\pi \in X_0(\kappa_n)$ is a geometric point lying over it:

We won’t denote the subscript $n$; hopefully it’s not too confusing.

This forms a part of sheaf-function correspondence alluded to by Bhargav.

Definition 4.6. For any functions $f, g : X_0(\kappa_n) \rightarrow \mathbb{C}$, we define a scalar product and its associated norm:

$$(f, g)_n = \sum_{y \in X_0(\kappa_n)} f(y) \overline{g(y)}, \quad \|f\|_n^2 = (f, f)_n.$$

Note that for the first sum to make sense, we think of $X_0(\kappa_n)$ as living inside $X_0(k)$.

These will give the space of functions $X_0(\kappa_n) \rightarrow \mathbb{C}$ an “$L^2$-structure,” which will be used to define the Fourier transform.

Now recall from last time that Brandon rewrote the logarithmic derivative of the $L$-function as follows:

$$\frac{\tau L'(X_0, \mathcal{G}_0, t)}{\tau L(X_0, \mathcal{G}_0, t)} = \sum_{x \in |X_0|} \sum_{n=1}^{\infty} d(x) \tau \text{Tr}(F_x^{n/d(x)}) t^{n-1} = \sum_{n=1}^{\infty} \left( \sum_{x \in |X_0|} d(x) \tau \text{Tr}(F_x^{n/d(x)}) \right) t^{n-1}.$$

Now since

$$(f_{\mathcal{G}_0}^n, 1)_n = \sum_{y \in X_0(\kappa_n)} f_{\mathcal{G}_0}^n(y) = \sum_{y \in X_0(\kappa_n)} \tau \text{Tr}(F_x^{n/d(x)}) = \sum_{x \in |X_0|} d(x) \tau \text{Tr}(F_x^{n/d(x)})$$
by using that a closed point in \(x\) with \(d(x)\) such that \(n\) corresponds to a \(\text{Gal}(\kappa_\nu/\kappa)-\text{orbit}\) in \(X_0(\kappa_\nu)\), we can rewrite the logarithmic derivative of the \(L\)-function as

\[
\frac{\tau L'(X_0, g_0, t)}{\tau L(X_0, g_0, t)} = \sum_{n=1}^{\infty} (f^{g_0}_n, 1)_n t^{n-1}.
\]

We will today look at a similar power series, except with the “\(L\)-radius of Convergence”

Theorem 4.1. The content of the following theorem is that we sometimes get the opposite inequality.

Definition 4.7. We define

\[
\phi^{g_0}(t) = \sum_{n=1}^{\infty} \left| f^{g_0}_n \right|^2 t^{n-1} = \sum_{n=1}^{\infty} \left( \sum_{x \in X_0(\kappa_\nu)} \left| \tau \text{Tr} \left( F_n^{d(x)} \right) \right|^2 \right) t^{n-1}.
\]

A possible reason for introducing \(\phi^{g_0}(t)\) is because it might work better with the Fourier transform, which will come later. As a bonus, its coefficients are nonnegative, and so it’ll be easier to compute its radius of convergence.

We want to show properties of \(\phi^{g_0}(t)\) as we did for the \(L\)-function last time:

Lemma 4.8. There is a constant \(C\) independent from \(n\) such that

\[
\left| f^{g_0}_n \right|^2 \leq C \cdot q^{n \cdot w(g_0) + \dim(X_0)}
\]

for all \(n \in \mathbb{Z}_{\geq 0}\), so \(\phi^{g_0}(t)\) converges for \(|t| < q^{-w(g_0) - \dim(X_0)}\).

Proof. We proceed as in Lemma 3.7. First,

\[
\left| f^{g_0}_n(x) \right|^2 = \left| \tau \text{Tr} \left( F_n^{d(x)} \right) \right|^2 \leq r^2 \cdot q^{n \cdot w(g_0)} \quad \text{where} \quad r = \max_{x \in |X_0|} \dim_{Q_\nu} g_0 x
\]

which we see is independent of \(x\), and so

\[
\left| f^{g_0}_n \right|^2 = \sum_{x \in X_0(\kappa_\nu)} \left| f^{g_0}_n(x) \right|^2 \leq \#X_0(\kappa_\nu) \cdot r^2 \cdot q^{n \cdot w(g_0)} \leq C \cdot q^{n \cdot (w(g_0) + \dim(X_0))},
\]

where as before, the last inequality is by the Noether normalization theorem.

This tells us a lower bound for the radius of convergence of \(\phi^{g_0}(t)\), but begs the question of whether \(q^{-w(g_0) - \dim(X_0)}\) is exactly the radius of convergence. Our main result is that this in fact is the radius of convergence, in some very nice cases.

Before we state the result, we introduce one more piece of notation:

Definition 4.9. We define the \(L^2\)-norm of a sheaf \(g_0\) as

\[
\|g_0\| = \sup \left\{ \rho \left| \lim_{n} \sup \frac{\left| f^{g_0}_n \right|^2}{q^{n(d \dim(X_0))}} > 0 \right\}. \]

Key Observation 4.10. \(q^{-\|g_0\| - \dim(X_0)}\) is the radius of convergence of \(\phi^{g_0}(t)\).

Note that by our discussion of the radius of convergence above, we always have

\[
\|g_0\| \leq w(g_0).
\]

The content of the following theorem is that we sometimes get the opposite inequality.

Recall that the generic maximal weight \(w_{\text{gen}}(g_0) = w(j_0^*g_0)\) where \(j_0 : U_0 \hookrightarrow X_0\) is any open dense immersion of a smooth subscheme. This was well-defined by the theorem on semicontinuity of weights from last time.

Theorem 4.11 (Radius of Convergence). Let \(g_0\) be a \(\tau\)-mixed sheaf on an algebraic scheme \(X_0\) of dimension \(\dim X_0 \leq 1\). Let \(j_0 : U_0 \hookrightarrow X_0\) be the open subscheme of \(X_0\) consisting of all irreducible components with dimension \(= \dim X_0\). Then we have (with the convention \(w_{\text{gen}}(j_0^*(g_0)) = -\infty\) for \(U_0 = \emptyset\))
1. $\|\mathcal{G}_0\| = \max(w_{gen}(\mathcal{G}_0^0(\mathcal{G}_0)), w(\mathcal{G}_0) - 1)$

2. Assume $X_0$ to be a smooth curve. If $H^1_E(X, \mathcal{F}) = 0$ for all closed subsets $E$ of $X$, then

$$\|\mathcal{G}_0\| = w(\mathcal{G}_0).$$

Remark 4.12. Here, a curve means a one-dimensional scheme $X_0$ that has pure dimension 1.

Proof. First, (1) $\Rightarrow$ (2) follows since if $X_0$ is smooth, we have

$$\|\mathcal{G}_0\| = \max(w_{gen}(\mathcal{G}_0^0(\mathcal{G}_0)), w(\mathcal{G}_0) - 1) = \max(w(\mathcal{G}_0), w(\mathcal{G}_0) - 1) = w(\mathcal{G}_0).$$

The proof of (1) will boil down to computing the radius of convergence of the power series $\mathcal{G}_0^0(t)$.

As always, we can assume $X_0$ is reduced. We moreover claim that it suffices to consider when $X_0$ is connected. First, write

$$\mathcal{G}_0^0(t) = \sum_{W \subset X_0} \mathcal{G}_{0W}(t).$$

Since each $\mathcal{G}_{0W}(t)$ has non-negative coefficients, we have that the radius of convergence of $\mathcal{G}_0^0(t)$ is the minimum of the radii of convergence of $\mathcal{G}_{0W}(t)$, and so

$$\|\mathcal{G}_0\| = \max_{W \subset X_0} \{\|\mathcal{G}_{0W}\|\};$$

and (1) would follow from the connected case.

The rest of the proof is by some case work. We first prove the $\dim X_0 = 0$ case, then prove the case where $\mathcal{G}_0$ is a smooth and $\tau$-pure sheaf on a smooth affine curve $X_0$, then the same for $\tau$-mixed $\mathcal{G}_0$, and then finally the general case.

Note that if $X_0$ is smooth, then (1) just says $\|\mathcal{G}_0\| = w(\mathcal{G}_0)$. This is what we will show in the first three cases. Also, since we already have shown $\|\mathcal{G}_0\| \leq w(\mathcal{G}_0)$, it suffices to show the opposite inequality.

Case 1. $\dim X_0 = 0$.

Assume $X_0$ is connected as above, and let $s \in |X_0|$ be the unique point in $X_0$. Then, consider the stalk $V = \mathcal{G}_0$ of $\mathcal{G}_0$ as a $\mathbf{C}$-vector space via the isomorphism $\tau : \mathcal{G}_0 \overset{\sim}{\rightarrow} \mathcal{C}$. The Frobenius map $F_s : V \rightarrow V$ can then be represented by a complex matrix $A$ with entries in $\mathbf{C}$. Let $\overline{A}$ be the conjugate matrix. Then, the function

$$\det \left(1_V - A \otimes \overline{A}, t^{d(s)} \right)^{-1}$$

has logarithmic derivative

$$\sum_{n=1}^{\infty} d(s) \text{Tr}((A \otimes \overline{A})^n) t^{d(s)n-1} = \sum_{n=1}^{\infty} d(s)(\text{Tr}(A^n))^2 t^{d(s)n-1} = \sum_{n=1}^{\infty} \|f\mathcal{G}_0\|^2_n t^{n-1} = \phi(\mathcal{G}_0),$$

and so the radius of convergence for $\mathcal{G}_0^0(t)$ is at most

$$\min_{\alpha, \beta \text{ eigenvalues}} |\tau(\alpha)\tau(\beta)|^{-1/d(s)} = \min_{\alpha \text{ eigenvalues}} |\tau(\alpha)|^{-2/d(s)} = q^{-w(\mathcal{G}_0)}$$

since these are the values of $t$ for which $\det(1_V - A \otimes \overline{A}, t^{d(s)})^{-1}$ diverges. Thus, $\|\mathcal{G}_0\| \geq w(\mathcal{G}_0)$, and we conclude $\|\mathcal{G}_0\| = w(\mathcal{G}_0)$.

Case 2. $\mathcal{G}_0$ a smooth, $\tau$-pure sheaf of weight $\beta$ on a smooth affine curve $X_0$.

Assume $X_0$ is connected as before, and also assume $X_0$ is geometrically irreducible like in the proof of the “curve case” from last time. Assume also that $\mathcal{G}_0 \neq 0$, for otherwise there would be nothing to show. Now consider the “complex conjugate” sheaf

$$\overline{\mathcal{G}_0} = \mathcal{G}_0^\vee \otimes \mathcal{L}_{\tau^{-1}(q^\beta)}$$

17
introduced last time to show \( \mathcal{G}_0 \) is the direct summand of a real sheaf. Then, \( \mathcal{G}_0 \otimes \mathcal{F}_0 \) is \( \tau \)-real, and the power series \( \phi^{\mathcal{G}_0}(t) \) is the logarithmic derivative of the \( L \)-series

\[
\tau L(X_0, \mathcal{G}_0 \otimes \mathcal{F}_0, t) = \prod_{\tau \in [X_0]} \tau \det \left( 1 - F_x t^{d(x)}, \mathcal{G}_0 \otimes \mathcal{F}_0 \right)^{-1}
\]

\[
= \frac{\tau \det \left( 1 - F_t, H^1_{\tau}(X, \mathcal{G} \otimes \mathcal{F}) \right)}{\tau \det \left( 1 - F_t, H^2_{\tau}(X, \mathcal{G} \otimes \mathcal{F}) \right)},
\]

where the last equality is by the Grothendieck trace formula, since

\[
(f^{\mathcal{G}_0 \otimes \mathcal{F}_0}(\tau), 1)_n = \| f^{\mathcal{G}_0}(\tau) \|_n^2.
\]

To show that \( \phi^{\mathcal{G}_0} \) has the correct radius of convergence, we will use that \( \mathcal{G}_0 \otimes \mathcal{F}_0 \) is \( \tau \)-pure of weight 2\( \beta \) by assumption on \( \tau \)-purity.

1. Lemma 3.7 says that this \( L \)-function has no zeros or poles in the region \(|t| < q^{-\beta-1}\).

2. We now want to show where poles could live. Using the last expression as a rational function, the proof of the “curve case” of semicontinuity from last time shows the poles of the \( L \)-function written above are of the form \( \frac{1}{\alpha q} \) where \( \alpha \) is an eigenvalue of

\[
F_x \circ ((\mathcal{G}_0 \otimes \mathcal{F}_0) \pi(X, x))
\]

for some stalk. As before, \( \alpha^{d(x)} \) is then an eigenvalue of \( F_x \circ \mathcal{G}_0 \otimes \mathcal{F}_0 \), hence has

\[
|\tau \alpha^{d(x)}|^2 = q^{2d(x)\beta}
\]

by assumption on \( \tau \)-purity. We therefore have

\[
|\tau \left( \frac{1}{\alpha q} \right) | = q^{-\beta-1},
\]

and so any pole of \( \tau L(X_0, \mathcal{G}_0 \otimes \mathcal{F}_0, t) \) has norm \( q^{-\beta-1} \).

3. Each “local \( L \)-factor”

\[
\tau \det \left( 1 - F_x t^{d(x)}, \mathcal{G}_0 \otimes \mathcal{F}_0 \right)^{-1}
\]

is a power series with leading coefficient 1. First of all, since \( \mathcal{G}_0 \not= 0 \), each local \( L \)-factor has some poles. Next, by Lemma 3.7 each local \( L \)-factor has a radius of convergence at least \( q^{-w(\mathcal{G}_0)} = q^{-\beta} \), and their poles have norm \( q^{-\beta-1} \) by (2), and so each local \( L \)-factor has a radius of convergence \( q^{-\beta-1} \). Thus, their product has a radius of convergence \( \leq q^{-\beta-1} \) (this uses some general facts about power series [KW01, Remark 2.17]).

This shows the radius of convergence of \( \phi^{\mathcal{G}_0} \) is \( q^{-\beta-1} = q^{-\|\mathcal{G}_0\|^{-1}} \), and so \( \|\mathcal{G}_0\| = w(\mathcal{G}_0) \).

**Case 3.** \( \mathcal{G}_0 \) a smooth, \( \tau \)-mixed sheaf on a smooth affine curve \( X_0 \).

Consider the filtration

\[
0 = \mathcal{G}_0^{(0)} \subset \mathcal{G}_0^{(1)} \subset \cdots \subset \mathcal{G}_0^{(r)} = \mathcal{G}_0
\]

where \( \mathcal{G}_0^{(j)} / \mathcal{G}_0^{(j-1)} \) is \( \tau \)-pure of weight \( \beta_j \). We would want to just use how traces interact with filtrations, but the issue is that the coefficients of \( \phi^{\mathcal{G}_0}(t) \) are a bit harder to work with because they are squares of things. Regardless, we can replace \( \mathcal{G}_0 \) by its “semisimplification”

\[
\bigoplus_j \mathcal{G}_0^{(j)} / \mathcal{G}_0^{(j-1)} =: \mathcal{F}_0 \oplus \mathcal{H}_0
\]

where \( \mathcal{F}_0 \) is the direct sum of all summands that are \( \tau \)-pure of weight \( w(\mathcal{H}_0) \), since this will not change the traces involved in the definition of \( \phi^{\mathcal{G}_0}(t) \). Note \( w(\mathcal{H}_0) < w(\mathcal{F}_0) \) by assumption.

Now since \( f^{\mathcal{G}_0} = f^{\mathcal{F}_0} + f^{\mathcal{H}_0} \) implies \( \| f^{\mathcal{G}_0} \|_n^2 = \| f^{\mathcal{F}_0} \|_n^2 + 2 \text{Re}(f^{\mathcal{F}_0}, f^{\mathcal{H}_0})_n + \| f^{\mathcal{H}_0} \|_n^2 \), we obtain

\[
\phi^{\mathcal{G}_0}(t) = \phi^{\mathcal{F}_0}(t) + \sum_{n=1}^{\infty} 2 \text{Re}(f^{\mathcal{F}_0}, f^{\mathcal{H}_0})_n t^{n-1} + \sum_{n=1}^{\infty} \| f^{\mathcal{H}_0} \|_n^2 t^{n-1}.
\]

18
1. The first term has radius of convergence \( q^{-w(\mathcal{F}_0)} - 1 \).
2. Lemma 4.8 implies the last term has radius of convergence at least \( q^{-w(\mathcal{H}_0)} - 1 \).
3. The inequality
   \[
   \text{Re}(f_{\mathcal{F}_0}, f_{\mathcal{H}_0}) - 1 \leq 2\|f_{\mathcal{F}_0}\| \cdot \|f_{\mathcal{H}_0}\| - 1 \leq C \cdot q^n \left( \frac{w(\mathcal{H}_0) + w(\mathcal{F}_0) + 1}{2} \right)
   \]
   again from Lemma 4.8 implies the second term has radius of convergence at least \( q^{-\left( \frac{w(\mathcal{H}_0) + w(\mathcal{F}_0) + 1}{2} \right)} \).

Thus, \( \phi_{\mathcal{F}_0}(t) \) has radius of convergence \( q^{-w(\mathcal{F}_0)} - 1 \), so \( w(\mathcal{F}_0) = \|\mathcal{F}_0\| \) as required.

**Case 4.** The general case.

This is the first time the term \( w_{\text{gen}}(j_0^*(\mathcal{H}_0)) \) in the maximum will appear.

Recall \( X_0 \) can be assumed to be reduced. In this case, we can find an open affine smooth curve (note curve means of pure dimension 1)

\[
h_0: V_0 \hookrightarrow X_0
\]
such that the complement

\[
i_0: S_0 \hookrightarrow X_0
\]
of \( V_0 \) is finite and such that \( h_0^*(\mathcal{F}_0) = \mathcal{F}_0 \) is smooth on \( V_0 \). Put also \( \mathcal{H}_0 = i_0^*(\mathcal{H}_0) \).

Now consider \( j_0: U_0 \hookrightarrow X_0 \) and the sheaf \( j_0^*(\mathcal{H}_0) \) on \( U_0 \). From definition of \( U_0 \), we have \( V_0 \subset U_0 \) and

\[
w_{\text{gen}}(j_0^*(\mathcal{H}_0)) = w(\mathcal{F}_0)
\]
by the semicontinuity theorem from last time. Then,

\[
\max\{w(\mathcal{F}_0), w(\mathcal{H}_0)\} = w(\mathcal{F}_0),
\]

\[
\max\{w(\mathcal{F}_0), w(\mathcal{H}_0) - 1\} = \max\{w_{\text{gen}}(j_0^*(\mathcal{H}_0)), w(\mathcal{F}_0) - 1\}.
\]

where the first equality is by definition of maximal weights, and the second is by considering how \( w(\mathcal{F}_0) \) and \( w(\mathcal{H}_0) - 1 \) compare.

Now use the short exact sequence

\[
0 \rightarrow h_0^!(\mathcal{F}_0) \rightarrow \mathcal{F}_0 \rightarrow i_0^*(\mathcal{H}_0) \rightarrow 0
\]
to obtain

\[
\phi_{\mathcal{F}_0}(t) = \phi_{\mathcal{F}_0}(t) + \phi_{\mathcal{H}_0}(t)
\]
by considering stalks. Now the coefficients of these power series are nonnegative so the radius of convergence of \( \phi_{\mathcal{F}_0} \) is the minimum of the radii of convergence of \( \phi_{\mathcal{F}_0} \) and \( \phi_{\mathcal{H}_0} \), which are \( q^{-w(\mathcal{F}_0)} - 1 \) and \( q^{-w(\mathcal{H}_0)} \), respectively. Thus,

\[
\|\mathcal{F}_0\| = \max\{w(\mathcal{F}_0), w(\mathcal{H}_0) - 1\} = \max\{w_{\text{gen}}(j_0^*(\mathcal{H}_0)), w(\mathcal{F}_0) - 1\}.
\]

\[ \square \]

### 5 May 25—Monodromy (Emanuel Reinecke)

**Notation 5.1.** \( X_0 \) denotes a geometrically connected, normal scheme of finite type over \( \kappa \); \( \pi \) denotes a geometric point; \( k \) denotes the algebraic closure of \( \kappa \); \( X \) denotes the base change of \( X_0 \) to \( \pi \); and \( \tau: \overline{\mathbb{Q}}_l \sim \mathbb{C} \)

is a fixed isomorphism.

Last two times, we defined pure and mixed sheaves, and weights on them. The problem with this, however, is that the definition for weights so far only make sense for mixed sheaves, and we don't know how to define them for general sheaves.

Today’s goal is to introduce a new notion of weights called “determinant weights.” These will be defined for all lisse sheaves, and will recover the standard notion for \( \tau \)-real sheaves (which we recall are sheaves whose characteristic polynomials of \( F_0 \) are real polynomials for all closed points \( a \in |X_0| \)).

The advantage of using determinant weights is that they are fairly easy to compute, and behave well with the usual cadre of operations (pullback, tensor product, exterior powers...).
We remind the reader of the following diagram from Tyler’s lecture, which we referred to as the monodromy exact sequence:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X, \mathfrak{r}) & \longrightarrow & \pi_1(X_0, \mathfrak{r}) & \longrightarrow & \text{Gal}(k/\kappa) & \cong & \mathbb{Z} & \longrightarrow & 1 \\
 & & \cup & & \cup & & & & \\
1 & \longrightarrow & \pi_1(X, \mathfrak{r}) & \longrightarrow & W(X_0, \mathfrak{r}) & \longrightarrow & W(k/\kappa) & \cong & \mathbb{Z} & \longrightarrow & 1 \\
\end{array}
\]  
(6)

Recall also that a lisse (Weil) sheaf \( \mathcal{G}_0 \) corresponds by Tannakian duality to a finite dimensional representation of \( W(X_0, \mathfrak{r}) \); thus, a rank 1 lisse sheaf corresponds to a character of \( W(X_0, \mathfrak{r}) \).

## 5.1 Determinant Weights

### 5.1.1 Rank 1 lisse sheaves

We first consider rank 1 (Weil) sheaves, which we can describe explicitly: they are all \( \tau \)-pure.

Let \( \mathcal{G}_0 \) be a lisse (smooth) sheaf of rank 1 on \( X_0 \). Let \( \chi : W(X_0, \mathfrak{r}) \to \mathbb{Q}_\ell^\times \) be the corresponding character. Our first main theorem is the following:

**Theorem 5.2.** If \( \chi : W(X_0, \mathfrak{r}) \to \mathbb{Q}_\ell^\times \) is a continuous character, then the image of \( \pi_1(X, \mathfrak{r}) \) via \( \chi \) in \( \mathbb{Q}_\ell^\times \) is finite.

We can restate this result in the following ways:

**Corollary 5.3.** There is a positive integer \( M \) such that \( \chi^M|_{\pi_1(X, \mathfrak{r})} \) is the trivial map.

**Corollary 5.4.** We can write \( \chi = \chi_1 \cdot \chi_2 \), where \( \chi_1 \) is torsion, and we have a factorization

\[
\begin{array}{ccc}
W(X_0, \mathfrak{r}) & \xrightarrow{\chi_2} & \mathbb{Q}_\ell^\times \\
\downarrow & \swarrow & \\
W(k/\kappa) & & \\
\end{array}
\]

**Proof.** The short exact sequence in (6) splits, since we can choose an arbitrary preimage of the generator \( \mathbb{Z} \) (note, however, that the splitting is not canonical). Thus, \( W(X_0, \mathfrak{r}) \) is a semidirect product, and we have the required decomposition.

**Corollary 5.5.** \( \mathcal{G}_0 \cong \mathcal{F}_0 \otimes \mathcal{L}_b \), where \( \mathcal{F}_0 \) is torsion and \( \mathcal{L}_b \) is pulled back from \( \kappa \). In particular, \( \mathcal{G}_0 \) is \( \tau \)-pure.

**Proof.** By Tannakian duality, Corollary 5.4 implies \( \mathcal{G}_0 = \mathcal{F}_0 \otimes \mathcal{L}_b \) for some \( b \in \mathbb{Q}_\ell^\times \), where

- \( \mathcal{F}_0 \) is pure of weight 0;
- \( \mathcal{L}_b \) (either as a sheaf on \( \text{Spec} \kappa \) or \( X_0 \)) has weight

\[
\frac{\log(|\tau(b)|)^2}{\log(q)}. 
\]

(7)

Thus, \( \mathcal{G}_0 \) is \( \tau \)-pure of weight as in (7).

**Remark 5.6.** Note that [KW01] demand that \( X_0 \) is geometrically irreducible, and in fact switch back and forth between assuming geometric irreducibility and geometric connectivity. But these notions are equivalent under our standing assumptions in Notation 5.1: \( \kappa \) is perfect, and \( X_0 \) normal, and so \( X = X_0 \otimes k \) is a base extension to a separable field extension, hence is still normal. Under these hypotheses, geometric connectivity implies geometric irreducibility.

We now give a sketch of the proof of Theorem 5.2.
Proof Sketch. By dévissage, we can reduce to the case where $X_0$ is a smooth, projective, geometrically connected curve (we will see this step later), and so we first consider this special case. We will show that we have a commutative diagram:

$$
\begin{array}{cccc}
\pi_1(X, \mathfrak{P}) & \longrightarrow & W(X_0, \mathfrak{P}) & \longrightarrow & \mathbb{Q}_\ell^* \\
\downarrow & & \downarrow & & \\
\text{Pic}^0(X_0)(\kappa) & \longrightarrow & W(X_0, \mathfrak{P})_{\text{ab}} & & \\
\end{array}
$$

(8)

Since $\text{Pic}^0(X_0)(\kappa)$ has only finitely many points (since $\text{Pic}^0(X_0) = \text{Jac}(X_0)$ is a projective variety), this would imply the theorem.

Let $K = K(X_0)$ be the function field of $X_0$. Then, we have a surjection

$$G_K = \pi_1(K) \twoheadrightarrow \pi_1(X_0, \mathfrak{P}) = G^\text{ur}_K,$$

where the latter group is the Galois group of the maximal Galois extension of $K$ that is everywhere unramified; since the pullback of a connected finite étale cover is connected, the connectivity criterion implies that the map above is surjective.

We now look at the abelianization of the diagram (6), and describe the elements of the top row below using class field theory for function fields:

$$
\begin{array}{cccc}
0 & \longrightarrow & \text{Pic}^0(X_0)(\kappa) & \longrightarrow & \text{Pic}(X_0)(\kappa) & \longrightarrow & 0 \\
D \rightarrow & & \bigg\uparrow & & \bigg\uparrow & & \\
0 & \longrightarrow & K^* \left( \frac{\langle A^*_K \rangle}{\prod_v \mathcal{O}_v^*} \right) & \longrightarrow & K^* \left( \frac{\langle A^*_K \rangle}{\prod_v \mathcal{O}_v^*} \right) & \longrightarrow & \mathbb{Q}^* \\
D \rightarrow & & \bigg\uparrow & & \bigg\uparrow & & \\
0 & \longrightarrow & I_K & \longrightarrow & W(X_0, \mathfrak{P})_{\text{ab}} & \longrightarrow & 0 \\
\pi_1(X, \mathfrak{P})_{\text{ab}} & \longrightarrow & \pi_1(X_0, \mathfrak{P})_{\text{ab}} & \longrightarrow & \text{Gal}(k/\kappa) \cong \hat{\mathbb{Z}} & \longrightarrow & 0 \\
\end{array}
$$

Note $\pi_1(X_0, \mathfrak{P})_{\text{ab}} = G^\text{ur,ab}_K$, the Galois group of the maximal everywhere unramified abelian extension of $K(X_0)$, and $I_K$ is the kernel of the degree map $W(X_0, \mathfrak{P})_{\text{ab}} \to W(k/\kappa) \cong \mathbb{Z}$. Also, $\langle A^*_K \rangle^1$ denotes the elements of $A^*_K$ with norm 1. By commutativity, the commutative diagram (8) follows.

We now give an idea for how to perform the dévissage reduction alluded to at the beginning of our proof. Assume for simplicity that $X_0 \subset \mathbb{P}^N$ is smooth, projective, geometrically connected, and of dimension $\geq 2$; note that the projectivity can be assumed by a theorem of Grothendieck, which says that replacing $\mathcal{G}_0$ by $\mathcal{G}^M$ for some $M > 0$, we can assume that $\mathcal{G}_0$ extends to the projective closure of (a suitable open subset of) $X$. Then, Bertini’s theorem implies that for a general linear subspace $L \subset \mathbb{P}^N$ of codimension $(\dim X - 1)$, the intersection $C := X \cap L$ is a smooth irreducible curve. After base field extension (which is okay since we may replace $\chi$ by $\chi^m$), we can assume $L$ and $C$ are defined over $\kappa$, and so $L = L_0 \otimes_k k$ and $C = C_0 \otimes_k k$. Assuming (after possible conjugation of $\pi_1(X, \mathfrak{P})$) that $\mathfrak{P} \in C$, we apply the connectivity criterion again to conclude $\pi_1(C, \mathfrak{P}) \twoheadrightarrow \pi_1(X, \mathfrak{P})$. \hfill \Box

Remark 5.7. Theorem 5.2 is not true if $X$ is not normal. Take $\mathbb{P}^1$ and glue $0$ to $\infty$ to form a nodal cubic $X$. We can take a trivial sheaf on $\mathbb{P}^1_\kappa$ and glue the fibres over $0$ and $\infty$ by any $\alpha \in \mathbb{Z}^*_\ell$. Then the image of the geometric fundamental group is $\alpha \hat{\mathbb{Z}}$, which is usually infinite.

Remark 5.8. We have $I_K \cong \pi_1(X, \mathfrak{P})_{\text{ab}}/(F - \text{id})\pi_1(X, \mathfrak{P})_{\text{ab}}$, and so we are using the fact that the Frobenius $F$ does not have a 1-eigenspace on $\pi_1(X, \mathfrak{P})_{\text{ab}}$. Roughly speaking, we are using that $\pi_1(X, \mathfrak{P})_{\text{ab}}$ does not have a weight zero component.
5.1.2 Determinant weights in general

Now we consider sheaves \( \mathcal{G}_0 \) of higher rank. Consider a composition series

\[ 0 = \mathcal{G}_0^{(0)} \subset \mathcal{G}_0^{(1)} \subset \cdots \subset \mathcal{G}_0^{(r)} = \mathcal{G}_0, \]

where \( \mathcal{F}_0^{(i)} := \mathcal{G}_0^{(i)}/\mathcal{G}_0^{(i-1)} \) are irreducible constituents of rank \( r_i \). Note such a composition series exists since \( \mathcal{G}_0 \) corresponds by Tannakian duality to a finite dimensional representation. Then, Proposition 5.5 implies that \( \bigwedge^{r_i} \mathcal{F}_0^{(i)} \) is \( \tau \)-pure.

**Definition 5.9.** The determinant weights of \( \mathcal{G}_0 \) are

\[ \frac{1}{r_i} \cdot \text{w} \left( \bigwedge^{r_i} \mathcal{F}_0^{(i)} \right). \]

5.2 Monodromy

5.2.1 Remarks on semidirect products

The material in this section was spread throughout the talk. We will be using basic properties of semidirect products, so we list them here. We omit proofs for the easy ones.

Let \( G \) and \( H \) be two groups. Recall that the outer isomorphism group \( \text{Out}(H) \) is the quotient of the automorphism group \( \text{Aut}(H) \) by the inner automorphisms.

**Proposition 5.10.** Let \( \alpha : G \to \text{Aut}(H) \) and \( \beta : G \to \text{Aut}(H) \) be two actions of \( G \) on \( H \). Then the two sequences

\[ 1 \to H \to G \rtimes_{\alpha} H \to G \to 1 \]

and

\[ 1 \to H \to G \rtimes_{\beta} H \to G \to 1 \]

are isomorphic if and only if the composite maps \( G \to \text{Out}(H) \) are equal.

**Proposition 5.11.** Let \( \rho : G \times H \to \text{GL}(V) \) be a representation of \( G \times H \). Then, \( G \times H \) permutes the \( H \)-subrepresentations of \( V \) in a containment-preserving way.

**Proposition 5.12.** Let \( \rho : G \times H \to \text{GL}(V) \) be a semisimple representation of \( G \times H \). Then \( \rho|_H \) is semisimple.

**Proof.** We immediately reduce to the case that \( \rho \) is a simple \( G \times H \) representation. An \( H \)-subrepresentation of \( V \) is simple and only if it has no proper \( H \)-subrepresentations, so \( G \times H \) preserves the set of simple \( H \)-subrepresentations.

Let \( W \) be the span of the simple \( H \)-subrepresentations. Then, \( W \) is a \( G \times H \) subrepresentation of \( V \), and \( V \) is simple, so \( V = W \). This shows that \( W \) is semi-simple as an \( H \)-representation.

5.2.2 Group theory of monodromy

Once again, let \( \mathcal{G}_0 \) be a lisse sheaf on \( X_0 \). Let \( \rho : W(X_0, \mathcal{F}) \to \text{GL}(V) \) be the corresponding representation, where \( V := \mathcal{G}_0^{(\pi)} \) is a \( \mathcal{Q}_\ell \)-vector space, which is finite-dimensional since \( \mathcal{G}_0 \) is lisse. Note that by definition of a \( \mathcal{Q}_\ell \)-sheaf, \( \rho \) is in fact defined over some finite extension \( E \) such that \( \mathcal{Q}_\ell \subset E \subset \mathcal{Q}_\ell \), and so \( V = W \otimes_E \mathcal{Q}_\ell \).

**Definition 5.13.** The images \( \rho(\pi_1(X, \mathcal{F})) \) and \( \rho(W(X_0, \mathcal{F})) \) in \( \text{GL}(W) \) are called the geometric monodromy group and the arithmetic monodromy group, respectively.

**Recollections 5.14.** Let \( G \) denote a group scheme defined over a field \( K \), where \( \text{Char} \ K = 0 \). Recall the following definitions and facts from representation theory:

1. \( G \) is a **linear algebraic group** if there exists a closed embedding \( G \hookrightarrow \text{GL}_n \); equivalently, \( G/K \) is of finite type and affine. \( G \) is automatically smooth since \( \text{Char} \ K = 0 \).

**Examples 5.15.** \( \text{GL}_n \) is trivially a linear algebraic group. A **torus** \( T/K \) (i.e., an algebraic group \( T \) such that \( T/K \cong G^r_m \) for some \( r \)) is also.
2. $G$ is (linearly) reductive if $\text{Rep}(G)$ is semisimple (this definition only works in characteristic zero). Alternatively (and this is the definition used in all characteristics), recall that $U \subset G$ is unipotent if there exists a closed embedding

$$U \hookrightarrow U_n = \left\{ (a_{ij}) \in \text{GL}_n \mid a_{ii} = 1 \quad a_{ij} = 0 \text{ for all } i > j \right\},$$

and then the unipotent radical $R_u(G)$ is defined to be the maximal connected, smooth, normal, unipotent subgroup of $G$. Then, one can show that $G$ is reductive if and only if $R_u(G) = \{1\}$.

**Examples 5.16.** $\text{GL}_n$ and tori $T$ are reductive.

**Fact 5.17.** If $G$ is a connected, smooth, commutative linear algebraic group, then $G \cong U \times T$ where $U$ is unipotent and $T$ is a torus. In particular, if $G$ is also reductive, then $G$ is a torus.

$R_u(-)$ commutes with finite field extensions $K \subset L$, that is, $R_u(G_L) = R_u(G)_L$.

3. The radical $R(G)$ of $G$ is the maximal normal, smooth, connected, solvable subgroup of $G$. $R(-)$ commutes with finite field extensions $K \subset L$, that is, $R(G_L) = R(G)_L$. $G$ is called semisimple if $R(G_{\overline{\kappa}}) = \{1\}$. Observe that if $G$ is semisimple, then it is reductive, since $R(G) \supset R_u(G)$.

**Example 5.18.** If $T$ is a torus, then $R(T_{\overline{\kappa}}) = T_{\overline{\kappa}}$.

**Fact 5.19.** If $G$ is a connected, reductive linear algebraic group over a perfect field $K$, then $R(G) = Z(G)_{\text{red}}$, the maximal central torus, where $H^\circ$ denotes the identity component of $H$. In particular, $G$ is semisimple if and only if $Z(G)$ is finite.

4. By Fact 5.17, if $G$ is a reductive group, then the center $Z(G)$ is the product of a torus and a finite group, and we have a short exact sequence

$$1 \rightarrow Z(G) \rightarrow G \rightarrow G_{\text{adj}} \rightarrow 1.$$

The same Fact shows that, again assuming $G$ is reductive, the abelianization $G_{\text{ab}}$ is the product of a torus and a finite group, and we have another short exact sequence:

$$1 \rightarrow [G, G] \rightarrow G \rightarrow G_{\text{ab}} \rightarrow 1.$$

Here, the groups $G_{\text{adj}}$ and $[G, G]$ are semisimple, and $[G, G] \rightarrow G_{\text{adj}}$ is an isogeny. In particular, $G$ is semisimple if and only if $Z(G)$ is finite (cf. Fact 5.19), if and only if $G_{\text{ab}}$ is finite.

**Example 5.20.** If $G = \text{GL}_n$, then the first sequence is

$$1 \rightarrow G_m \rightarrow \text{GL}_n \rightarrow \text{PGL}_n \rightarrow 1,$$

and the second is

$$1 \rightarrow \text{SL}_n \rightarrow \text{GL}_n \rightarrow G_m \rightarrow 1.$$

We now return to our setting. Let $\mathscr{G}_0$ be a lisse sheaf, and denote $V = \mathscr{G}_{\overline{\kappa}}$ to be a stalk of $\mathscr{G}$ at a geometric point $\overline{\kappa}$. Recall $\rho: W(X_0, \overline{\kappa}) \rightarrow \text{GL}(V)$ denotes the representation of $W(X_0, \overline{\kappa})$ corresponding to $\mathscr{G}_0$ via Tannakian duality.

**Notation 5.21.** We denote $G_{\text{geom}}$ to be the Zariski closure of $\rho(\pi_1(X, \overline{\kappa}))$ in $\text{GL}(V)$, that is, it is the smallest Zariski-closed subgroup of $\text{GL}(V)$ whose $\overline{\mathbb{Q}}_\ell$-points contain $\rho(\pi_1(X, \overline{\kappa}))$. $G_{\text{geom}}$ is a linear algebraic group over $\overline{\mathbb{Q}}_\ell$.

We note from the proof of Corollary 5.4 that the short exact sequence

$$1 \rightarrow \pi_1(X, \overline{\kappa}) \rightarrow W(X_0, \overline{\kappa}) \rightarrow W(k/\kappa) \cong \mathbb{Z} \rightarrow 1$$

23
is split (non-canonically). In particular, we obtain an action of $\mathbb{Z}$ on $\pi_1(X, \mathfrak{p})$, canonical up to inner automorphism. Applying $\rho$ to the first two terms of this short exact sequence, we get a commutative diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(X, \mathfrak{p}) & \longrightarrow & W(X_0, \mathfrak{p}) & \longrightarrow & W(k/\kappa) & \longrightarrow & 1 \\
\downarrow & & \downarrow \rho & & \downarrow & & \\
1 & \longrightarrow & G_{\text{geom}} & \longrightarrow & GL(V) & & \\
\end{array}
$$

by definition of $G_{\text{geom}}$.

Now choose some $g \in W(X_0, \mathfrak{p})$ lifting $1 \in \mathbb{Z}$. Note via the identification $W(k/\kappa) = \langle F \rangle \cong \mathbb{Z}$, this element $g \in W(X_0, \mathfrak{p})$ lifts the Frobenius $F \in W(k/\kappa)$. Then, $\rho(g)$ normalizes $\rho(\pi_1(X, \mathfrak{p}))$, that is, $\rho(g)\rho(\pi_1(X, \mathfrak{p}))\rho(g^{-1}) = \rho(\pi_1(X, \mathfrak{p}))$, hence $\rho(g)$ normalizes the Zariski closure $G_{\text{geom}}$ of $\rho(\pi_1(X, \mathfrak{p}))$ by continuity. This induces an action of $W(k/\kappa) = \langle F \rangle \cong G_{\text{geom}}$ via conjugation.

**Definition 5.22.** We denote $G_{\text{arith}}$ to be $G_{\text{geom}} \times W(k/\kappa) \cong G_{\text{geom}} \times \mathbb{Z}$, where the action of $\mathbb{Z}$ is via $1 \mapsto \rho(g)$ as above.

Note that the presentation of $G_{\text{arith}}$ as a semidirect product depends on the choice of lifting, but the group $G_{\text{arith}}$ does not. So we have a semidirect short exact sequence in the middle row below, such that the entire diagram commutes:

$$
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(X, \mathfrak{p}) & \longrightarrow & W(X_0, \mathfrak{p}) & \longrightarrow & W(k/\kappa) & \longrightarrow & 1 \\
\downarrow & & \downarrow \rho & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G_{\text{geom}} & \longrightarrow & G_{\text{arith}} & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
& & \downarrow \deg & & & & \downarrow & & \\
& & GL(V) & & & & & & \\
\end{array}
$$

(9)

Let us now assume $\mathcal{G}_0$ is semi-simple (for example, we may have started with a general lisse sheaf and then taken one of its composition factors). By the Tannakian duality between local systems and representations of $\pi_1$, this means that $\rho: \pi_1(X, \mathfrak{p}) \times \mathbb{Z} \cong W(X_0, \mathfrak{p}) \to GL(V)$ is semisimple. By Proposition 5.12, $\rho|_{\pi_1(X, \mathfrak{p})}$ is semisimple. This implies that $G_{\text{geom}}$ is reductive by Recollection 5.14(3), i.e., it has no normal unipotent subgroups.

We will show later that $G_{\text{geom}}$ is semisimple, which means that all occurrences of the groups $Z(G_{\text{geom}})$ and $(G_{\text{geom}})_{\text{ab}}$ below will be finite by Recollection 5.14(4). However, we first want the following result:

**Lemma 5.23.** There is a positive integer $N$ such that the semidirect sequence

$$
1 \longrightarrow G_{\text{geom}} \longrightarrow \text{deg}^{-1}(N \cdot \mathbb{Z}) \xrightarrow{\text{deg}} N \cdot \mathbb{Z} \longrightarrow 1
$$

from (9) is direct, in other words, $\text{deg}^{-1}(N \cdot \mathbb{Z}) \cong G_{\text{geom}} \times \mathbb{Z}$.

Once again, let $g \in W(X_0, \mathfrak{p})$ be a chosen lift of $1 \in W(k/\kappa) \cong \mathbb{Z}$.

**Proof.** We set $G = G_{\text{geom}}$ for brevity. The representation $\rho: Z(G) \to GL(V)$ contains finitely many characters $\chi_1, \chi_2, \ldots, \chi_s$. Then, $\mathbb{Z}$ acts on $Z(G)$ by conjugation by $\rho(g) \in GL(V)$, and thus must permute this list of characters. Replacing $g$ by a power of $g$, we may assume this permutation is trivial, so that the action of $\mathbb{Z}$ on $Z(G)$ will be trivial.

Also, $g$ acts on $G_{\text{adj}}$. The outer automorphism group of a semisimple group is the automorphism group of the Dynkin diagram, and thus likewise finite. So, replacing $g$ by a power of $g$, we may assume that the action of $g$ on $G_{\text{adj}}$ is inner and, changing the choice of semidirect splitting, we may assume the action on $G_{\text{adj}}$ is trivial.

After these reductions, the action on $G$ is of the form $(\begin{smallmatrix} \eta & \eta \\ 0 & 0 \end{smallmatrix})$ for some $\eta \in \text{Hom}(G_{\text{adj}}, Z(G))$. But there are no nontrivial maps from a connected semisimple group to a torus, so any map $G_{\text{adj}} \to Z(G)$ is trivial on the connected component of the identity, and we see that $\text{Hom}(G_{\text{adj}}, Z(G))$ is likewise finite.

Passing to one more power of $g$, the action on $G$ is now trivial and the product is direct. 

□
We can now show the following:

**Theorem 5.24** (Grothendieck). Let $\mathcal{G}_0$ be geometrically semisimple and lisse. Then,
1. $G_{\text{geom}}$ and $G_{\text{geom}}^0$ are semisimple;
2. Denoting $Z := Z(G_{\text{arith}}(\overline{Q}_l))$, the map $Z \xrightarrow{\psi} W(k/\kappa)$ has finite kernel and cokernel. More precisely, $Z$ has a power of an element of degree 1.

**Corollary 5.25.** After base field extension, $Z \xrightarrow{\psi} W(k/\kappa)$ is surjective.

**Proof of Theorem 5.24.** For (1), choose $N$ as in Lemma 5.23 replacing $\kappa$ by its degree $N$ extension, we can assume $N = 1$. So we have the following commutative diagram, where the second row is split by Lemma 5.23, $\pi$ is the projection onto $G_{\text{geom}}$, and the dashed arrow is $\pi \circ \rho$:

\[
\begin{array}{c}
1 & \longrightarrow & \pi_1(X, \pi) & \longrightarrow & W(X_0, \pi) & \longrightarrow & Z & \longrightarrow & 1 \\
1 & \longrightarrow & G_{\text{geom}} & \longrightarrow & G_{\text{geom}} \times Z & \longrightarrow & Z & \longrightarrow & 1 \\
& & \downarrow^{\alpha} & & \downarrow^{\rho} & & \downarrow & & \downarrow \\
& & (G_{\text{geom}})_{\text{ab}} & & & & & & \\
\end{array}
\]

Recall that $(G_{\text{geom}})_{\text{ab}}$ is the product of a finite group and a torus, and so applying Theorem 5.2 to $\alpha \circ \pi \circ \rho$, the image of $\pi_1(X, \pi)$ in $(G_{\text{geom}})_{\text{ab}}$ is finite. But it must also be Zariski dense by definition of $G_{\text{geom}}$. So $(G_{\text{geom}})_{\text{ab}}$ is finite and $G_{\text{geom}}$ is semisimple. Passing to the identity component, we see that $G_{\text{geom}}^0$ is also semisimple.

For (2), $\ker \psi \subset Z(G_{\text{geom}}(\overline{Q}_l))$ is finite since $G_{\text{geom}}$ is semisimple. For the cokernel, we want to show that there exists $z \in Z(G_{\text{arith}}(\overline{Q}_l))$ such that $\deg(z) \neq 0$. As in (1), we can assume $N = 1$ in Lemma 5.23, and so there exists $\zeta \in G_{\text{arith}}(\overline{Q}_l)$ such that $\deg \zeta = 1$ and $\zeta \in C(G_{\text{geom}}(\overline{Q}_l))$, the centralized of $G_{\text{geom}}(\overline{Q}_l)$ in $G_{\text{arith}}$. Now for $g \in G_{\text{geom}}(\overline{Q}_l)$, define a cocycle

$$\phi: Z \longrightarrow G_{\text{geom}}(\overline{Q}_l)$$

$$n \longmapsto g \cdot \zeta^n \cdot g^{-1} \zeta^{-n}$$

This is a homomorphism since

$$\phi_g(n + m) = \phi_g(n) \zeta^n \phi_g(m) \zeta^{-n} = \phi_g(n) \phi_g(m),$$

where the first equality is by definition and the second is since $\zeta \in C(G_{\text{geom}}(\overline{Q}_l))$. Also, one can check that

$$g' \cdot \phi_g(n) g^{-1} = \phi_g(n)$$

for all $g \in G_{\text{geom}}(\overline{Q}_l)$ and $g' \in G_{\text{geom}}(\overline{Q}_l)$. This implies $\im \phi_g \subset Z(G_{\text{geom}}(\overline{Q}_l))$, which is finite since $G_{\text{geom}}$ is semisimple, and so there there exists $n > 0$ such that $\phi_g(n) = 1$ for all $g \in G_{\text{geom}}(\overline{Q}_l)$. Thus, $\zeta^n \in C(G_{\text{geom}}(\overline{Q}_l))$, and $\zeta^n \in Z(G_{\text{arith}}(\overline{Q}_l))$ since $G_{\text{arith}}(\overline{Q}_l)$ is generated by $\zeta$ and $G_{\text{geom}}(\overline{Q}_l)$. Finally, setting $z := \zeta^n$ gives us our desired element. 

### 5.3 Applications

**Lemma 5.26.** Let $\mathcal{G}_0$ be semisimple and lisse, and let $z \in Z(G_{\text{arith}}(\overline{Q}_l))$ with $\deg(z) = n \neq 0$, as in the proof of Theorem 5.24(2). Suppose $z$ acts on $V = \mathcal{G}_0^r$ with eigenvalues $\alpha_1, \ldots, \alpha_r$, where

$$|\tau(\alpha_i)^2| = q^{n \beta_i}.$$ 

Then, $\beta_1, \ldots, \beta_r$ are determinant weights of $\mathcal{G}_0$. 

25
Proof. $z \in Z(G_{\text{arith}}(\mathcal{O}_k))$ implies that $z$ is a homomorphism of $G$-modules. Then, by Schur’s lemma, $\rho(z)$ acts by eigenvalues on irreducible summands, and so reducing to the case when $\mathcal{G}_0$ is irreducible, we have that $z$ acts by some eigenvalue $\alpha$. So $z$ acts on $\bigwedge^{\dim V} V$ by $\alpha^{\dim V}$. Looking at the definition of determinant weights, we have $q^{n \beta} = |\tau(\alpha)^2|$. \hfill $\square$

We also conclude that determinant weights behave in the obvious way for pullback, direct sum, and tensor product, by considering how the eigenvalues of $z$ change for these operations:

**Theorem 5.27.** Let $f_0: X'_0 \to X_0$ be a dominant morphism between normal, geometrically connected schemes, and let $\mathcal{F}_0$ and $\mathcal{G}_0$ be lisse sheaves on $X_0$. Then,

1. $\mathcal{G}_0, f'^*\mathcal{G}_0$ have the same determinant weights;
2. If $\mathcal{F}_0, \mathcal{G}_0$ each have single determinant weights $\alpha, \beta$, then $\mathcal{F}_0 \otimes \mathcal{G}_0$ has determinant weight $\alpha + \beta$;
3. For all weights $\gamma$, denoting $r(\gamma)$ to be the sum of ranks of irreducible constituents with weight $\gamma$, the determinant weights of $\bigwedge^r \mathcal{F}_0$ are
   \[\sum_{\gamma \in \mathbb{R}} n(\gamma) \cdot \gamma, \quad \text{where} \quad \sum n(\gamma) = r, \; 0 \leq n(\gamma) \leq r(\gamma), \; \text{and} \; n(\gamma) \in \mathbb{Z}.\]

Proof of (2) and (3). Replace $\mathcal{F}_0, \mathcal{G}_0$ with their semisimplifications, and use Lemma 5.26 \hfill $\square$

6 May 30—Real Sheaves (Matt Stevenson)

**Notation 6.1.** Unless otherwise stated, $\kappa = \mathbb{F}_q, k = \mathbb{F}$; $X_0$ is an algebraic (i.e., finite type) scheme over $\kappa$; $\mathcal{G}_0$ is a (Weil) sheaf on $X_0$; $\tau: \mathcal{O}_X \to \mathbb{C}$ is a fixed isomorphism; and if $x \in |X_0|$ is a closed point, then $\tau$ is a geometric point over $x$, $d(x) = [\kappa(x) : \kappa]$, and $N(x) = \# \kappa(x) = q^{d(x)}$.

We recall something Brandon talked about a while ago (see Definition 3.12). Recall that $\mathcal{G}_0$ is $\tau$-real if for all $x \in |X_0|$, the characteristic polynomial of geometric Frobenius $F_x: \mathcal{G}_0\tau \to \mathcal{G}_0\tau$ has real coefficients, i.e., if $\tau \det(1 - F_x t, \mathcal{G}_0\tau) \in \mathbb{R}[t]$.

**Goal 6.2.** We want to show $\tau$-real sheaves are in fact $\tau$-mixed, and that the determinant weights of a $\tau$-real sheaf are the $\tau$-weights.

We will do this for $X_0, \mathcal{G}_0$ satisfying some additional hypotheses.

6.1 Curve case

Let $X_0$ be a smooth geometrically irreducible curve over $\kappa$, and let $\mathcal{G}_0$ be a smooth sheaf on $X_0$. Then, the Weil group $W(X_0, \tau)$ acts on $V = \mathcal{G}_0\tau$, and $\pi := \pi_1(X, \mathbb{X}) \subset W(X_0, \overline{\mathbb{X}})$.

**Remark 6.3.** $H^2(X, \mathcal{G}) = V_\tau(-1)$ since they are Poincaré dual to $H^0(X, \mathcal{G}) = V^\tau$, and so if $\alpha$ is an eigenvalue of Frobenius $F \circ H^2(X, \mathcal{G})$, then $\alpha q^{-1}$ is an eigenvalue of $F \circ V_\tau$. Thus,

$$\log \left( \frac{|\tau(\alpha q^{-1})|^2}{\log(q)} \right)$$

is a determinant weight of $\mathcal{F}_0 = \overline{V}_\tau$ on Spec $\kappa$. By Theorem 5.27, this is a determinant weight of $\mathcal{F}_0'$, the pullback of $\mathcal{F}_0$ on $X_0$, hence is also a determinant weight of $\mathcal{G}_0$.

This is saying that if we have an idea for what eigenvalues of the Frobenius look like, then we also have an idea for what determinant weights look like.

**Remark 6.4.** If $\mathcal{G}_0$ is $\tau$-real, then the logarithmic derivative of $\tau \det(1 - F_x t, \mathcal{G}_0\otimes k)^{-1}$ is

$$f(t) = \sum_{n=1}^{\infty} \tau(\text{Tr}(A^n)k) \cdot t^{n-1},$$

26
where $A$ is the matrix of the Frobenius on $\mathcal{G}_0$. Since $\mathcal{G}_0$ is $\tau$-real, all of these $\tau\left(\text{Tr}(A^n)^k\right)$ are real, and so the power series
\[
\tau \det(1 - F_xt, \mathcal{G}_0^{\otimes k})^{-1} = e^{\int f(t) \, dt}
\]
has real coefficients, so $\mathcal{G}_0^{\otimes k}$ is $\tau$-real for all $k \geq 1$. In particular, the coefficients of $\tau \det\left(1 - F_xt, \mathcal{G}_0^{\otimes 2k}\right)^{-1}$ are in $\mathbb{R}_{\geq 0}$.

We can now talk about our main theorem in the curve case.

**Theorem 6.5** (Rankin–Selberg Method). Let $X_0$ be a smooth, geometrically irreducible curve over $\kappa$, and $\mathcal{G}_0$ be a smooth sheaf over $X_0$ that is $\tau$-real. Then, all irreducible constituents of $\mathcal{G}_0$ are $\tau$-pure, and their $\tau$-weights coincide with their determinant weights.

**Remark 6.6.** Deligne does not refer to this result as the Rankin–Selberg method. Katz mentions this result as Rankin’s method in [Kat01, Introduction]. Laumon may also refer to this as Rankin’s method.

We first need an a priori estimate for eigenvalues of Frobenius.

**Lemma 6.7.** Let $X_0, \mathcal{G}_0$ be as in Theorem 6.3, let $\alpha$ be an eigenvalue of $F_x \otimes \mathcal{G}_0$, and let $\beta$ be the largest determinant weight of $\mathcal{G}_0$. Then,
\[
|\tau(\alpha)|^2 \leq N(x)^\beta.
\]

**Proof.** Assume $\mathcal{G}_0 \neq 0$, and $X_0$ is affine (so that $H^0(X, \mathcal{G}) = 0$). Then, for all $k \geq 1$, $2k\beta$ is the largest determinant weight of $\mathcal{G}_0^{\otimes 2k}$ by Theorem 5.27. If $t_0$ is a zero of $\tau \det(1 - F_xt, H^1_c(X, \mathcal{G}^{\otimes 2k}))$, then
\[
\log \left(\frac{|t_0^{-1}q^{-1}|^2}{\log(q)}\right)
\]
is a determinant weight of $\mathcal{G}_0^{\otimes 2k}$, hence $\leq 2k\beta$. Equivalently, $|t_0| \geq q^{-(2k\beta+2)/2}$.

Now we write down the Grothendieck trace formula as a quotient of two characteristic polynomials; this will let us determine when the infinite product may or may not converge.

For $k \geq 1$, the Grothendieck trace formula for $\mathcal{G}^{\otimes 2k}$ says
\[
\prod_{x \in |X_0|} \tau \det(1 - F_xt^{d(x)}, \mathcal{G}_0^{\otimes 2k})^{-1} = \frac{\tau \det(1 - F_t, H^1_c(X, \mathcal{G}^{\otimes 2k}))}{\tau \det(1 - F_t, H^2_c(X, \mathcal{G}^{\otimes 2k}))}.
\]
The denominator on the right-hand side has no zeros in the disk $|t| < q^{-(2k\beta+2)/2}$, and so the infinite product on the left-hand side converges there. Thus, each of the “local $L$-factors” also converge there (since if you have a complex power series which is a product of power series with leading term one, with non-negative real coefficients (using that our power $2k$ is even), then the radius of convergence of the entire product is at most the radius of convergence of each factor). In particular, these polynomials $\tau \det(1 - F_xt^{d(x)}, \mathcal{G}_0^{\otimes 2k})$ are zero-free in the region $|t| < q^{-(2k\beta+2)/2}$.

Now given an eigenvalue $\alpha$ of $F_x \otimes \mathcal{G}_0$, its $(2k)$th power $\alpha^{2k}$ is an eigenvalue of $F_x \otimes \mathcal{G}_0^{\otimes 2k}$. So, this formula tells us that
\[
|\tau(\alpha^{-2k/d(x)})| \geq q^{-(2k\beta+2)/2}
\]
Equivalently,
\[
|\tau(\alpha)|^2 \leq q^{d(x)(2k\beta+2)/2k} = N(x)^{\beta + \frac{1}{k}}.
\]
Finally, send $k \to +\infty$. ∎

We are now ready to prove the Theorem.

**Proof of Theorem 6.5.** For $\beta \in \mathbb{R}$, we will denote $\mathcal{G}_0(\beta)$ to be the direct sum of irreducible constituents of $\mathcal{G}_0$ with determinant weight $\beta$, and denote $r(\beta) = \text{rank} \mathcal{G}_0(\beta)$. Now for a fixed closed point $x \in |X_0|$, the eigenvalues of $\mathcal{G}_0$ will appear as eigenvalues of some $\mathcal{G}_0(\beta)$, so let
\[
\alpha_{1}^\beta, \ldots, \alpha_{r(\beta)}^\beta
\]
be the eigenvalues of $F_x \otimes \mathcal{G}_0(\beta)$. We want to show that $|\tau(\alpha_i^\beta)|^2 = N(x)^{\beta}$ for all $i = 1, \ldots, r(\beta)$. We would want to just apply Lemma 6.7 to both $\mathcal{G}_0$ and its dual $\mathcal{G}_0^\vee$ to get inequalities in both directions, but it turns out we need to modify $\mathcal{G}_0$ a bit first for this to actually be possible.
Observation 6.8. The determinant weight $\beta$ of $\mathcal{G}_0(\beta)$ can be written in terms of the $\alpha_j^\beta$, by definition of determinant weights:

$$\beta = \frac{1}{r(\beta)} \sum_{j=1}^{r(\beta)} \log \left( \left| \tau(\alpha_j^\beta) \right|^2 \right).$$

Equivalently,

$$\left| \tau \left( \prod_{j=1}^{r(\beta)} \alpha_j^\beta \right) \right|^2 = N(x)^\beta r(\beta).$$

Now, letting $N = \sum_{\gamma > \beta} r(\gamma)$ (which is a finite sum since there are only finitely many determinant weights in $\mathcal{G}_0$), we know by Theorem 5.27(3) what the determinant weights of $\wedge^{N+1} \mathcal{G}_0$ look like:

- $\mathcal{G}_0 \wedge \left( \wedge^N \mathcal{G}_0 \right) = \wedge^{N+1} \mathcal{G}_0$ has largest determinant weight

$$\beta + \sum_{\gamma > \beta} r(\gamma) \cdot \gamma.$$

We can get many more determinant weights by lowering $r(\gamma)$.

- An eigenvalue of $F_x$ on $\wedge^{N+1} \mathcal{G}_0$ is

$$\alpha_i^\beta \prod_{\gamma > \beta} \alpha_j^\gamma.$$

Now (10) tells us that

$$\left| \tau \left( \alpha_i^\beta \prod_{\gamma > \beta} \prod_{j=1}^{r(\gamma)} \alpha_j^\gamma \right) \right|^2 \leq N(x)^\beta + \sum_{\gamma > \beta} r(\gamma) \cdot \gamma.$$

Observation 6.8 implies that, by pulling the multiplication out,

$$\left| \tau \left( \prod_{\gamma > \beta} \prod_{j=1}^{r(\gamma)} \alpha_j^\gamma \right) \right|^2 = \prod_{\gamma > \beta} \left| \tau \left( \prod_{j=1}^{r(\gamma)} \alpha_j^\gamma \right) \right|^2 = \prod_{\gamma > \beta} N(x)^{r(\gamma) \cdot \gamma}.$$

Thus, $|\tau(\alpha_i^\beta)|^2 \leq N(x)^\beta$.

Now we can get the opposite inequality by replacing $\mathcal{G}$ with its dual $\mathcal{G}^\vee$. 

\[ \square \]

6.2 General Case

Theorem 6.9. Let $X_0$ be an algebraic scheme, $\mathcal{G}_0$ be a $\tau$-real sheaf. Then,

1. $\mathcal{G}_0$ is $\tau$-mixed;
2. Purity Let $X_0$ be irreducible and normal, and let $\mathcal{G}_0$ be smooth. Then, the irreducible constituents of $\mathcal{G}_0$ are $\tau$-pure of the appropriate weight (that is, equal to their determinant weight).

Proof. (2) follows from (1) and Theorem 3.10(3). This isn’t completely formal since you need to understand the proof of (1) to get (2).

For (1), first we note the following:

Observation 6.10. Let $j_0: U_0 \hookrightarrow X_0$ be open, and let $i_0: S_0 \hookrightarrow X_0$ be the closed complement of $U_0$. Then, we have a short exact sequence

$$0 \rightarrow j_{0*} \mathcal{G}_0 \rightarrow \mathcal{G}_0 \rightarrow i_{0*} i_0^* \mathcal{G}_0 \rightarrow 0,$$

and so it suffices to show $j_{0*} \mathcal{G}_0$ and $i_0^* \mathcal{G}_0$ are $\tau$-mixed. Since we have already shown the dimension 1 case in Theorem 6.5 we know $i_0^* \mathcal{G}_0$ is $\tau$-mixed already by noetherian induction.
Remark 6.11. We claim we can freely pass to finite field extensions of our base field. Let \( \kappa'/\kappa \) be a finite extension, and write \( X'_0 = X_0 \otimes_\kappa \kappa' \). If \( \mathcal{F}_0 \), the pullback of \( \mathcal{F}_0 \) to \( X'_0 \), is \( \tau \)-mixed, then its direct image in \( X_0 \) will be \( \tau \)-mixed, and \( \mathcal{F}_0 \) will be a subsheaf of this direct image, hence will also be \( \tau \)-mixed (this uses facts from Permanence Properties 3.4).

These two facts (and reductions similar to what we have been doing so far; see [KW01, §I.3]) allow us to reduce to the following case: \( X_0 \) is smooth, irreducible, and affine; \( \mathcal{F}_0 \) is smooth; \( \dim X_0 > 1 \); and all irreducible constituents of \( \mathcal{F}_0 \) are geometrically irreducible.

Now we can embed \( X_0 \) into some projective space \( \mathbb{P}^N_0 \) over \( \kappa \), and let \( \mathcal{F}_0 \) be an irreducible constituent of \( \mathcal{F}_0 \). We want to show that we can find an open subset \( U_0 \subset X_\kappa \) such that \( \mathcal{F}|_{U_0} \) is \( \tau \)-pure.

The idea is as follows. We use the geometric irreducibility assumption to be able to base change to the algebraic closure \( k \), and use (a version of) Bertini’s theorem. This allows us to find an open set \( U \subset X_\kappa \) where (1) holds, and then we can hope to descend this \( U \) to an open set defined over \( \kappa \).

Consider linear subspaces \( L \subset \mathbb{P}^N_0 \) (over \( k = \kappa \)) of codimension \( \dim X_0 - 1 \), so that \( X \cap L \) is (generically) a curve. These linear subspaces will be the \( k \)-points of some Grassmannian \( G \). We can consider those particular \( L \)'s that satisfy some nice properties: \( L \cap X =: C \) is a nonempty, smooth, irreducible curve; and \( \mathcal{F}|_C \) is irreducible. The Bertini theorem for Weil sheaves (see [KW01, Thm. B.1]) says these \( L \)'s are the \( k \)-points of a nonempty open subset \( \Omega \subset G \). And for such an \( L \in \Omega \), there exists a finite field extension \( \kappa'/\kappa \) such that \( C \) is defined over \( \kappa' \), i.e., there exists \( C_0 \subset X_0 \otimes_\kappa \kappa' \) a closed curve such that \( C = C_0 \otimes_\kappa \kappa' \).

Let \( \mathcal{F}_0' \) be the pullback of \( \mathcal{F}_0 \) to \( C_0 \), and let \( \mathcal{F}_0'' \) be the pullback of \( \mathcal{F}_0 \) to \( C_0 \). Then, \( \mathcal{F}_0'' \) is still \( \tau \)-real, \( \mathcal{F}_0' \) is still irreducible, and \( \mathcal{F}_0'' \) is an irreducible constituent of \( \mathcal{F}_0'' \). Now the Rankin–Selberg method (Theorem 6.5) says that \( \mathcal{F}_0'' \) is \( \tau \)-pure of the appropriate weight.

To finish our proof of the theorem, we allow \( L \in \Omega \) to vary. There exists a nonempty open \( U \subset X_\kappa \) such that every \( k \)-point of \( U \) is contained in at least one of these \( L \in \Omega \). We can assume that \( U \) is defined over \( \kappa \) since it is defined over some finite extension of \( \kappa \), and so you can replace \( U \) by the intersection of its Galois conjugates to get an open subset defined over \( \kappa \). Then we get a subset \( U_0 \subset X_0 \) such that its base change is an open subset \( U \) as above (which might be smaller than the original \( U \)).

Remark 6.12. It may be possible to avoid passing to the algebraic closure \( k \) by using Poonen’s Bertini theorems over finite fields.

### 7 June 1—Fourier Transforms (Charlotte Chan)

We will first be very concrete by reviewing Fourier transforms of functions defined over finite fields.

The classical Fourier transform is defined over \( \mathbb{R} \), using integrals. The analogue for finite fields \( \mathbb{F}_q \) is instead defined by sums.

**Notation 7.1.** We will fix for today a non-trivial (i.e., not always equal to 1) additive character

\[
\psi: \mathbb{F}_q \rightarrow \mathbb{C}^*.
\]

Note that this induces characters on every finite extension \( \mathbb{F}_{q^n} \) by

\[
\psi: \mathbb{F}_{q^n} \xrightarrow{\text{Tr}} \mathbb{F}_q \xrightarrow{\psi} \mathbb{C}^*,
\]

which we will also denote by \( \psi \).

We will fix an isomorphism \( \tau: \overline{\mathbb{Q}}_q \xrightarrow{\sim} \mathbb{C} \), and for today will suppress this notation so we don’t get bogged down in notation. In addition, as always, \( \kappa_0 = \mathbb{F}_q \), \( \kappa_n = \mathbb{F}_{q^n} \), and \( k = \overline{\mathbb{F}}_q \).

We start by recalling the definition of the Fourier transform for functions defined over finite fields.

**Definition 7.2** (Fourier transform over finite fields). Let \( f: \mathbb{F}_{q^n} \rightarrow \mathbb{C} \). Then, the Fourier transform of \( f \) is defined as the function

\[
T_\psi f: \mathbb{F}_{q^n} \rightarrow \mathbb{C}
\]

\[
x \mapsto \sum_{y \in \mathbb{F}_{q^n}} f(y)\psi(-xy).
\] (11)
As in the real case, we have the following formulas:

**Theorem 7.3** (Plancherel Formula). \( \|T_\psi f\|^2_n = q^n \|f\|^2_n \), where we recall
\[
\|f\|^2_n = (f,f)_n = \sum_{x \in F_q^n} f(x)\overline{f(x)}.
\]

**Theorem 7.4** (Fourier Inversion). \( T_{\psi^{-1}} T_\psi f = q^n f \).

Our goal is to use the “sheaf-to-functions” correspondence to develop an analogue of the Fourier transform for complexes of sheaves. More precisely:

**Goal 7.5.** Letting \( K_0 \in D^b_c(A^1_0, \mathbb{Q}_l) \), we want to construct another complex \( T_\psi K_0 \) whose corresponding function satisfies Theorems 7.3 and 7.4.

Recall that a subscript 0 denotes that the object in question is defined over \( \kappa = F_q \).

We start by recalling the sheaf-to-functions correspondence.

**Recall 7.6** (Sheaf-to-functions correspondence). To \( K_0 \in D^b_c(A^1_0, \mathbb{Q}_l) \), we associate the function
\[
f_{K_0} : F_q^n \rightarrow \mathbb{C}
\]
\[
x \mapsto \sum_i (-1)^i \mathrm{Tr}(F_{q^n}, \mathcal{H}^i_c(K_0)).
\]

To get analogues of Theorems 7.3 and 7.4, what we will do is develop proofs side by side with the proofs of the results for functions to get an idea for how to prove the sheaf-theoretic versions.

**Recollections 7.7.** Before we begin, we recall the following facts about constructible sheaves:

1. Grothendieck trace formula: For a constructible sheaf \( \mathcal{F}/X_0 \), we have
\[
\sum_{x \in X_0(F_q^n)} \mathrm{Tr}(F_x, \mathcal{F}) = \sum_i (-1)^i \mathrm{Tr}(F_{q^n}, H^i_c(X, \mathcal{F})).
\]

2. Base change: Consider the following cartesian square of finite type schemes over \( F_q \):
\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow \scriptstyle{f'} & & \downarrow \scriptstyle{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

Then, “going along the dashed arrow in either direction gives the same result,” that is, there is a natural isomorphism of functors
\[ Rf^! \circ g^* \cong g^* \circ Rf_! \]

3. Special case of base change: If \( Y' = \{y\} \) is a point in \( Y \), then the cartesian square in (2):
\[
\begin{array}{ccc}
X_y & \xrightarrow{i} & X \\
\downarrow \scriptstyle{i_y} & & \downarrow \scriptstyle{f} \\
\{y\} & \xrightarrow{i_y} & Y
\end{array}
\]
gives, for a complex \( K_0 \) on \( X \),
\[ Rf^! i^* K_0 \cong i_y^* Rf_! K_0 \]
The stalk of \( Rf_! K_0 \) at \( y \) is equal to the stalk of \( Rf^! K_0 \) at \( y \), and so this gives the isomorphism
\[ \text{the stalk of } R^i f_! K_0 \text{ at } y \cong H^i_c(X_y, K_0). \]
4. Combining (1) and (3): For a morphism \(f: X \to Y\) and a complex \(K_0\) on \(X\),

\[
f^{Rf_! K_0}(y) = \sum_{x \in X_y(F_q^n)} f^{K_0}(x),
\]

where \(y\) is a \(F_q^n\)-point, by using the sheaf-to-functions correspondence.

5. We also translate (2) to the language of functions. If \(y' \in Y'\), then

\[
\sum_{x' \in X'} f^{K_0}(g'(x')) = \sum_{x \in X} g(y') f^{K_0}(x).
\]

6. Projection formula: If \(f: X \to Y\) is a morphism, and \(M, N\) are defined on \(X, Y\), respectively, then

\[
Rf_!(f^* N \otimes M) \cong N \otimes Rf_! M,
\]

and so pullback and pushforward have some compatibility. In terms of functions,

\[
\sum_{x \in X} y f^N(g(x)) f^M(x) = \sum_{x \in X} y f^N(f(x)) f^M(x).
\]

Note this is an example of where the result for functions is trivial, but the proof on the sheaf side is more complicated.

These facts will help us prove analogues of Theorems 7.3 and 7.4 for complexes; the sums on the function side will help us understand where the proofs come from.

We first need to define a sheaf associated to \(\psi\). To do so, we use Artin–Schreier sheaves. Consider the morphism \(\mathbb{P}: \mathbb{A}_0^1 \to \mathbb{A}_0^1\)

\[
x \mapsto x^q - x.
\]

This is a finite Galois étale covering, with Galois group \(F_q\), and so we get a surjection

\[
\pi_1(\mathbb{A}_0^1, \mathbb{A}_0^1) \to F_q.
\]

So, attached to \(\psi\), we get a rank one étale sheaf \(\mathcal{L}_0(\psi)\) called the Artin–Schreier sheaf on \(\mathbb{A}_0^1\), since we have a character

\[
\pi_1(\mathbb{A}_0^1, \mathbb{A}_0^1) \to F_q \to C^* \to C^*
\]

of the fundamental group.

**Remark 7.8.** If we base change \(\mathcal{L}_0(\psi)\) to \(F_q^n\), we get \(\mathcal{L}_0(\text{Tr} \circ \psi)\), that is

\[
\mathcal{L}_0(\psi) \otimes F_q^n = \mathcal{L}_0(\text{Tr} \circ \psi).
\]


Now that we have defined a sheaf, the logical thing to ask is what function this sheaf corresponds to.

**Lemma 7.9.** \(f^{Z_0(\psi)}(x) = \psi(-x)\). In particular, the weight \(w(Z_0(\psi)) = 0\).

Note that \(Z_0(\psi)\) is \(\tau\)-pure, so \(w(Z_0(\psi))\) is the unique weight of \(Z_0(\psi)\).

**Proof.** Let \(x \in F_q^n = \mathbb{A}_0^1 \otimes F_q^n\). Then, consider the arithmetic Frobenius

\[
\sigma: \overline{F}_q \to \overline{F}_q \to \overline{F}_q
\]

\[
\alpha \mapsto \alpha^{q^n}
\]

Then, if \(\alpha\) satisfies \(\alpha^{q^n} - \alpha = x\), then also \(\sigma(\alpha)^{q^n} - \alpha = x\). On the other hand, \(\sigma(\alpha) - \alpha = x\), and so \(\sigma(\alpha) = x + \alpha\), i.e., \(\sigma\) corresponds to addition by \(x\). Thus, the geometric Frobenius \(F_x\), which is the inverse of the arithmetic Frobenius, corresponds to subtraction by \(x\), that is, it translates \(\alpha \mapsto \alpha - x\). In particular, \(F_x\) acts on \(Z_0(\psi)\) by \(\psi(-x)\). \(\Box\)
We have a way of getting sums and products of functions via operations on the sheaf side from Recollections 7.7, and so we put them together to define an analogue of the Fourier transform in the sheaf case. Note that our naïve guess is not completely right and we need to introduce a shift of our complex.

**Notation 7.10.** Round brackets (1) denote Tate twists, that is, \( K(1) := K \otimes \mathbb{Q}_\ell(1) \), and square brackets \([1]\) denote the degree shift to the left for complexes, i.e., \( K[1] := K^{i+1} \).

**Definition 7.11.** Consider the following diagram

\[
\begin{array}{c}
\mathbb{A}^1_0 \times \mathbb{A}^1_0 \xrightarrow{m} \mathbb{A}^1_0 \rightarrow \mathcal{L}_0(\psi) \\
\pi^1 \downarrow \quad \pi^2 \downarrow \\
\mathbb{A}^1_0 \quad \quad \quad \mathbb{A}^1_0
\end{array}
\]

where \( m \) denotes the multiplication map \((x, y) \mapsto xy\). Consider the sheaf \( \mathcal{L}_0(\psi) \) defined on the codomain of \( m \), and \( K_0 \) defined on the codomain of \( \pi^2 \). Then, we define the functor Fourier transform

\[
T_\psi : \mathcal{D}_c^b(A^1_0, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_c^b(A^1_0, \mathbb{Q}_\ell)
\]

by

\[
T_\psi K_0 = R\pi_1!* (\pi^2_* K_0 \otimes m^* \mathcal{L}_0(\psi))[1].
\]

Later we will see why this shift \([1]\) is needed: it fixes the perversity.

This is a direct analogue of the Definition 7.2, except for the shift \([1]\). If we wanted to make Definition 7.2 match our new definition, we would need to introduce a minus sign at the front of (11):

**Lemma 7.12.** \( f^{T_\psi K_0}(x) = -\sum_{y \in \mathbb{F}_q^n} f^K(y) \psi(-xy) \) for \( x \in \mathbb{F}_q^n \).

**Proof.** By definition,

\[
f^{T_\psi K_0}(x) = \sum_i (-1)^i \text{Tr}(F_i, \mathcal{H}^{i}(\mathbb{F}_q^n \otimes m^* \mathcal{L}_0(\psi))[1])
\]

By pulling out the shift \([1]\) and using Recollection 7.7(4),

\[
= -\sum_i (-1)^i \sum_{y \in \mathbb{F}_q^n} \text{Tr}(F_i, \mathcal{H}^{i}(\mathbb{F}_q^n \otimes m^* \mathcal{L}_0(\psi)))
\]

\[
= -\sum_{y \in \mathbb{F}_q^n} \sum_i (-1)^i \text{Tr}(F_i, \mathcal{H}^{i}(K_0)) \cdot \psi(-xy)
\]

\[
= -\sum_{y \in \mathbb{F}_q^n} f^K(y) \cdot \psi(-xy).
\]

The following is very important, even if the proof is short.

**Theorem 7.13.** Let \( a \in \mathbb{F}_q \) be a geometric point of \( A^1 = A^1_0 \otimes \mathbb{F}_q \). Then,

\[
(T_\psi(K_0))_a = R\Gamma_c(K \otimes \mathcal{L}(\psi_a))[1],
\]

where \( \psi_a : \mathbb{F}_q^n \to \mathbb{C} \) maps \( x \mapsto \psi(ax) \), and \( \mathbb{F}_q^n \) is chosen such that it contains \( a \). The (complexes of) sheaves on the right are the pullbacks of those with subscripts \( 0 \) to \( A^1 = A^1_0 \otimes \mathbb{F}_q \).

**Proof.** Using Recollection 7.7(3) (base change) above,

\[
(T_\psi(K_0))_a = R\Gamma_c((\pi^2_* K \otimes m^* \mathcal{L}(\psi))[1]) = R\Gamma_c(K \otimes \mathcal{L}(\psi_a))[1].
\]
Remark 7.14. This Theorem is very important for the purposes of the seminar: recall that what we want to study is Frobenius actions on cohomology $H^i_c(A^1, K)$. Now, we’ve taken these cohomology spaces and stuck them into a nice family (a sheaf), so that we can view $T^\psi(K_0)$ as a “deformation” of $H^*_{c}(A^1, K)$. Taking stalks will eventually give the information we wanted:

$$\mathcal{H}^i((T^\psi(K_0))_0) = H^i_{c}(A^1, K),$$

since $R\Gamma_c(-)$ gives the complex used to compute cohomology. This will be used crucially in the next talk.

We are now ready to show the analogue of Theorem 7.3:

**Theorem 7.15 (Plancherel Formula).** $\|T^\psi(K_0)\|^2_n = q^{n/2} \|K_0\|^2_n$.

**Remark 7.16.** We have that, assuming $H^i(K_0)$ are all $\tau$-mixed,

$$w(K_0) = \max \{w(H^i(K_0)) - i \}.$$

Thus, the Plancherel formula above can be interpreted in terms of weights, using Theorem 4.11:

$$w(T^\psi(K_0)) = w(K_0) + 1.$$

**Proof of Theorem 7.15.** By definition and Lemma 7.12,

$$\|T^\psi(K_0)\|^2_n = \sum_{x \in \mathbb{F}_q^n} f^\psi(K_0)(x)f^\psi(K_0)(x) = \sum_{x,y,z \in \mathbb{F}_q^n} f^{K_0}(y)f^{K_0}(z)\psi(-xy)\psi(xz).$$

Now note that

$$\sum_x \psi(x(z-y)) = \begin{cases} 0 & \text{if } z \neq y \\ q^n & \text{if } z = y \end{cases} \quad (12)$$

and so

$$\|T^\psi(K_0)\|^2_n = \sum_{x,y,z \in \mathbb{F}_q^n} f^{K_0}(y)f^{K_0}(z)\psi(0) = q^n(f,f)_n = q^n\|f^{K_0}\|^2_n. \quad \square$$

We can rephrase the calculation (12) in terms of the Fourier transform of the constant function 1:

$$T^\psi(1) = \sum_{x \in \mathbb{F}_q^n} \psi(xy) = \begin{cases} 0 & \text{if } y \neq 0 \\ q^n & \text{if } y = 0 \end{cases}$$

This tells us what we should expect the Fourier transform of the constant sheaf $\mathcal{Q}_\ell$ should be:

**Lemma 7.17.** Let $\delta_0 := i_0_*\mathcal{Q}_\ell$ be the skyscraper sheaf, where $i_0 : \{0\} \hookrightarrow A^1$. Then,

$$T^\psi(\mathcal{Q}_\ell[1]) = \delta_0(-1).$$

**Proof of Lemma 7.17.** We use the following:

**Fact 7.18 ([KW01], p. 42).** Using the Leray spectral sequence for the Artin–Schreier cover $\varphi : x \mapsto x^q - x$, together with the fact that

$$\varphi_*\mathcal{Q}_\ell \cong \bigoplus_{x \in \mathbb{F}_q} \mathcal{L}(\psi_x),$$

one can show

$$H^i_c(A^1, \mathcal{L}(\psi_x)) = \begin{cases} \mathcal{Q}_\ell(-1) & x = 0, \ i = 2 \\ 0 & \text{else} \end{cases}$$

Now use base change (Recollection 7.73):

$$R\pi^*_1(m^*\mathcal{L}(\psi))_x[1] = R\Gamma_c(\mathcal{L}(\psi_x))[1] = \delta_0(-1)[-1]. \quad \square$$
Remark 7.19 (Perversity). Fix middle perversity. The sheaf $\mathcal{O}_i[1]$ is perverse and so is $\delta_0(-1)$, since we are pushing forward from a point. So the [1] is important to preserve perversity.

We will now prove the sheaf analogue of Fourier Inversion (Theorem 7.4), using the sum formulation as guidance.

Theorem 7.20 (Fourier Inversion). $T_{\psi^{-1}}T_{\psi}K_0 = K_0(-1)$

The Tate twist $(-1)$ multiplies the geometric Frobenius $F_{q^n}$ action by $q^n$, and this corresponds to the $q^n$ factor in the function case.

We will use base change and the projection formula repeatedly in the proof, so we will only refer to them by name, not by their references, i.e., Recollections 7.7(3) and (6).

Proof. We will develop the proof by writing down the proofs for sheaves and functions side-by-side; for convenience, on the function side we will write $f = f^{K_0}$. First, by definition

\[ T_{\psi^{-1}}T_{\psi}K_0 \]

\[ (T_{\psi^{-1}}T_{\psi}f)(x) = \mathbb{R}\pi_1^1(\pi^2_1(\mathbb{R}\pi_1^1(\pi^{2*}K_0 \otimes m^*\mathcal{L}(\psi)) \otimes m^*\mathcal{L}(\psi^{-1})) [2] = \sum_y \left( \sum_z f(z)\psi(-yz) \right) \psi(xy) \]

We want to remove the parentheses on the function side. To perform the corresponding operation on sheaves, we consider the following diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
A_0^1 & \times & A_0^1 & \times & A_0^1 \\
\pi^{12} & \circlearrowright & \pi^{23} & \circlearrowleft & \pi^2 \\
A_0^1 & \times & A_0^1 & \times & A_0^1 \\
\pi & \circlearrowright & \pi & \circlearrowleft & \pi \\
A_0^1 & \times & A_0^1 & \times & A_0^1 \\
\end{array}
\end{array}
\]

where the top square is cartesian. Now applying base change to this cartesian square, we can put all of our sheaves into one space:

\[ = \mathbb{R}\pi_1^1(\mathbb{R}\pi_1^{12}(\pi^{23*}K_0 \otimes m^{*}\mathcal{L}(\psi)) \otimes m^{*}\mathcal{L}(\psi^{-1})) [2] \]

and then use the projection formula to get

\[ = \mathbb{R}\pi_1^1 \circ \mathbb{R}\pi_1^{12}(\pi^{23*} \pi^{2*}K_0 \otimes \pi^{23*}m^{*}\mathcal{L}(\psi) \otimes \pi^{12*}m^{*}\mathcal{L}(\psi^{-1})) [2] \]

Next, we want to combine $\psi(-yz)\psi(xy) = \psi(-(yz - yx)) = \psi(-y(z - x))$. Let $\alpha: A^3 \to A^2$ be defined by $(x, y, z) \mapsto (y, z - x)$; then, we obtain

\[ = \mathbb{R}\pi_1^1 \circ \mathbb{R}\pi_1^{12}(\pi^{23*} \pi^{2*}K_0 \otimes \alpha^{*}m^{*}\mathcal{L}(\psi)) [2] \]

We now want to change the order of summation on the function side. We note $\mathbb{R}\pi_1^1 \circ \mathbb{R}\pi_1^{12} = \mathbb{R}\pi_1^1 \circ \mathbb{R}\pi_1^{13}$ and $\pi^{23*} \pi^{2*} = \pi^{13*} \pi^{2*}$ to obtain

\[ = \mathbb{R}\pi_1^1 \circ \mathbb{R}\pi_1^{13}(\pi^{13*} \pi^{2*}K_0 \otimes \alpha^{*}m^{*}\mathcal{L}(\psi)) [2] \]

By the projection formula, we can pull out $f(z)$ on the function side:

\[ = \mathbb{R}\pi_1^1(\pi^{2*}K_0 \otimes \mathbb{R}\pi_1^{13}\alpha^{*}m^{*}\mathcal{L}(\psi)) [2] = \sum_y \sum_z f(z)\psi(-y(z - x)) \]

34
Now consider the diagram

\[
\begin{array}{ccc}
(x,y,z) & \xymatrix{ \ar[r] & (y,z-x) } & A^3_0 \\
& \ar[d]^{z^3} & A^2 \\
A^2_0 & \ar[r]^{\beta} & A^1_0 \\
& \ar[d]^z & A^1_0 \\
(x,z) & \ar[r]^{\alpha} & z-x
\end{array}
\]

By base change with respect to this cartesian square, we obtain

\[
R\pi_1^!\left(\pi^{2*}K_0 \otimes \beta^* R\pi_2^!(m^*\mathcal{L}_0(\psi))\right)[2]
\]

which we can interpret on the functions side as thinking of \(z-x\) as the fixed quantity in the inner sum, instead of \(z\). By Lemma 7.17, we get the sheaf equivalent of (12) for functions:

\[
R\pi_1^!\left(\pi^{2*}K_0 \otimes \beta^* \delta_0(-1)[-2]\right)[2] = R\pi_1^!\left(\pi^{2*}K_0 \otimes \beta^* \delta_0(-1)\right) = \sum_{z} f(z) \cdot \begin{cases} q^n \text{ if } z-x = 0 \\ 0 \text{ else} \end{cases}
\]

Now applying base change with the cartesian square

\[
\begin{array}{ccc}
A^1_0 & \xymatrix{ \ar[r] & * } & A^1_0 \\
\Delta & \ar[d] \ar[r]^j & \ar[d] \ar[r]^{io} & \ar[r] & \\
A^2_0 & \beta & A^2_0
\end{array}
\]

we obtain

\[
R\pi_1^!\left(\pi^{2*}K_0 \otimes R\Delta_*\mathcal{Q}_\ell(-1)\right) = \sum_{z} f(z) \sum_{x \in z} q^n
\]

By using the projection formula again, we can pull out the factor of \(q^n\) on the function side to get:

\[
R\pi_1^! \circ R\Delta_* (\Delta^* \pi^{2*} K_0 \otimes \mathcal{Q}_\ell) (-1) = q^n \sum_{z \in F, z \neq x} f(z)
\]

Finally, by the commutativity of the diagram

\[
\begin{array}{ccc}
A^1_0 & \xymatrix{ \ar[r] & A^1_0 } & A^1_0 \\
\Delta & \ar[d]^{\pi^1} \ar[r]^\text{id} & \ar[d]^{\pi^2} \ar[r]^{\text{id}} & A^2_0 \\
A^2_0 & \ar[r]^\Delta & A^2_0 & \ar[r]^\beta & A^1_0
\end{array}
\]

we have \(R\pi_1^! \circ R\Delta_1 = R\text{id}_1 = \text{id}\) and \(\Delta^* \pi^{2*} = \text{id}^*\), i.e., they have no affect on sheaves, and so we obtain our desired result:

\[
K_0(-1) = q^n f(x).
\]

Even though the proof for sheaves was a bit long, notice that the proof on the function side tells you how you should proceed in the proof.

Finally, we can also define an analogue of the Fourier transform in higher dimensions, and we get similar results as before.
Definition 7.21 (Partial Fourier Transform). Fix our ground field $\mathbf{F}_q$. Given the “dot product” map

$$A_0^n \times A_0^n \xrightarrow{m} A_0^1$$

$$\langle x, y \rangle \mapsto \sum x_i y_i$$

write

$$A_0^n \times A_0^n \times A_0^{n-r} \times Y_0 \xrightarrow{m} A_0^1 \xrightarrow{\mathcal{L}_0(\psi)}$$

$$\xrightarrow{\pi^1} \xrightarrow{\pi^2} \xrightarrow{K_0}$$

Consider the sheaf $\mathcal{L}_0(\psi)$ defined on the codomain of $m$, and $K_0$ defined on the codomain of $\pi^2$. Then, we define the partial (functor) Fourier transform as

$$T_{\psi, r} K_0 = R^1 \pi_1^!(\pi^{2*} K_0 \otimes m^* \mathcal{L}_0(\psi)) \lceil r \rceil.$$

(14)

Remark 7.22. Fix middle perversity. We would like all operations in (14) to preserve perversity, and explain the shift $\lceil r \rceil$:

1. The functor $\pi^{2*}$ shifts perversity, so in order to preserve middle perversity we need to apply the shift $\lceil r \rceil$ by the codimension.
2. The functor $R^1 \pi_1^!$ is in general only left-t-exact. But Fourier inversion below will tell us that $R^1 \pi_1^!$ is right-t-exact as well.

Theorem 7.23 (Fourier Inversion). $T_{\psi^{-1}, r} T_{\psi, r} K_0 = K_0(-r)$.

Proof. Use base change and the projection formula to show

$$T_{\psi^{-1}, r} \circ T_{\psi, r} = T_{\psi^{-1}, r-1} \circ T_{\psi, r-1} \circ T_{\psi^{-1}, r} \circ T_{\psi, r},$$

and proceed by induction on $r$, where the base case is Theorem 7.20.

Theorem 7.24. $T_{\psi} : \text{Perv}(A_0^n \times Y_0, \mathbb{Q}_{\ell}) \rightarrow \text{Perv}(A_0^n \times Y_0, \mathbb{Q}_{\ell})$ is an equivalence.

Before giving all of the details, we give the idea for the proof. First, just by definition of $T_{\psi}$, we have that if $K_0 \in \text{Perv}(A_0^n \times Y, \mathbb{Q}_{\ell})$, then $T_{\psi} K_0 \in pD^{1,0}(A_0^n \times Y_0, \mathbb{Q}_{\ell})$. We then want to apply Fourier inversion (Theorem 7.23) to show $p_{\tau \geq 1} T_{\psi} K_0 = 0$.

Proof. By definition, we have

$$T_{\psi}(K_0) = R^1 \pi_1^!(\pi^{2*} K_0 \lceil r \rceil \otimes \pi^* \mathcal{L}_0(\psi)).$$

Note that, for the perverse t-structure (and not for the standard constructible t-structure),

1. $R^1 \pi_1^!$ is left t-exact since $\pi_1$ is affine;
2. $K_0 \mapsto \pi^{2*} K_0 \lceil r \rceil$ is t-exact since $\pi^2$ is equidimensional and smooth of fibre dimension $r$;
3. $- \otimes \pi^* \mathcal{L}_0(\psi)$ is t-exact since $\pi^* \mathcal{L}_0(\psi)$ is a local system.

Thus, $T_{\psi} K_0 \in pD^{1,0}(A_0^n \times Y_0, \mathbb{Q}_{\ell})$.

Now to show that $T_{\psi} K_0$ is actually perverse, we want to show that $p_{\tau \geq 1} T_{\psi}(K_0) = 0$. To do so, we just need to write down the correct distinguished triangles.

Consider the distinguished triangle

$$pH^0 T_{\psi} B \longrightarrow T_{\psi} B \longrightarrow p_{\tau \geq 1} T_{\psi} B \longrightarrow p_{\tau \leq 0} T_{\psi} B \tag{15}$$

Since $T_{\psi^{-1}}$ is exact, we can apply Fourier inversion (Theorem 7.23) to obtain a new distinguished triangle:

$$T_{\psi^{-1}} pH^0 T_{\psi} B \longrightarrow B(-r) \xrightarrow{0} T_{\psi^{-1}} p_{\tau \geq 1} T_{\psi} B \longrightarrow$$

36
where the map $B(-r) \to T_{\psi^{-1}} p_{\tau \geq 1} T_{\psi} B$ must be the zero map since $B(-r)$ is perverse. This implies that the map $T_{\psi} B \to p_{\tau \geq 1} T_{\psi} B$ in $[15]$ is also the zero map.

The rotation of $[15]$ is the distinguished triangle

$$T_{\psi} B \xrightarrow{0} \xrightarrow{p_{\tau \geq 1} T_{\psi} B} \xrightarrow{p H^0 T_{\psi} B[1]} \xrightarrow{\text{Cone}} T_{\psi} B[1] \oplus p_{\tau \geq 1} T_{\psi} B$$

Thus,

$$p H^0 T_{\psi} B \cong T_{\psi} B \oplus p_{\tau \geq 1} T_{\psi} B[-1].$$

Applying $p_{\tau \geq 1}$ to both sides of this isomorphism gives

$$p_{\tau \geq 1} p H^0 T_{\psi} B \cong p_{\tau \geq 1} T_{\psi} B \oplus p_{\tau \geq 1} p_{\tau \geq 1} T_{\psi} B[-1],$$

but the left-hand side is zero, and so we have that the right-hand side is also zero. In particular, $p_{\tau \geq 1} T_{\psi} B = 0$, and so $T_{\psi} B \in \text{Perv}(A_0^0 \times Y_0, \mathbb{Q}_l)$.

Finally, $T_{\psi} : \text{Perv}(A_0^0 \times Y_0) \to \text{Perv}(A_0^0 \times Y_0)$ is an equivalence since it has an inverse $T_{\psi^{-1}}(r)$. □

8 June 6—Weil Conjectures I (Bhargav Bhatt)

We start with a rough chart of the logical structure of the proof of the Weil conjectures so far:

![Chart](chart.png)

Today we will use these results to prove the Weil conjectures.

8.1 Curve case

Deligne’s theorem for curves is the following:

**Theorem 8.1.** Let $U_0$ be a smooth affine curve over $\mathbb{F}_q$, and let $F_0$ be a lisse $\overline{\mathbb{Q}}_l$-sheaf on $U_0$, which is $\tau$-mixed of weight $\leq w$. Then, $H^2_{\psi}(U, F) = \tau$-mixed of weight $\leq w + i$.

We start with a series of reductions to reduce to the case when $U_0 = \mathbb{A}^1_0$.

1. We may assume $i = 1$: $i = 0$ vanishes and the $i = 2$ case is obvious by Poincaré duality

   $$H^2_{\psi}(U, F) \cong H^0(U, F^\vee)^\vee(-1).$$

2. We may shrink $U_0$: If $j_0 : V_0 \hookrightarrow U_0$ is a dense open immersion, then the map $H^1_{\psi}(V, F|_{V}) \to H^1_{\psi}(U, F)$, obtained by covariant functoriality with respect to immersions, is surjective since the cokernel of $j_0! (F_0|_{V_0}) \to F_0$ has finite support.
3. We may extend the base field by finite extensions freely.
4. We may assume \( U_0 \subset \mathbf{A}_F^1 \): By noether normalization, you can find a diagram that looks like

\[
\begin{array}{c}
V_0 \\ \text{finite étale} \end{array} \longrightarrow \begin{array}{c} U_0 \\ \pi \downarrow \text{finite} \end{array} \quad \begin{array}{c} W_0 \\ \pi \downarrow \text{finite} \end{array} \longrightarrow \mathbf{A}_F^1
\]

and we can replace \((U_0, F_0)\) by \((W_0, \pi_*(F_0|_{V_0}))\), since \(H_c^1(V, F|_V) \cong H_c^1(W, \pi_*(F|_V))\). Note that we cannot control the rank of \( \pi_*(F_0|_{V_0}) \) in this process.
5. We may assume \( F_0 \) is (geometrically) irreducible: \( H_c^1(U, -) \) is a left-exact functor on local systems over \( U \), and so we can split up \( F_0 \) into its irreducible constituents, which become geometrically irreducible after a finite base extension.
6. We may assume that \( F_0 \) is unramified at infinity, that is, that \( F_0 \) extends to a local system on \( U_0 \cup \{ \infty \} \subset \mathbf{P}^1_{\mathbf{F}_q} \): Pick an unramified point \( u \) in \( U_0 \subset \mathbf{A}_F^1 \), shrink \( U_0 \) so \( u \not\in U_0 \), and move \( u \) to \( \infty \) by a Möbius transformation (after possibly extending the base field so that the transformation is defined over it).
7. We may assume \( F_0 \) is not (geometrically) constant, that is, not constant after passing to the algebraic closure \( \overline{\mathbf{F}}_q \) of our base field, by proving this case separately. Suppose \( F_0 \cong \mathbf{Q}_\ell \), and write \( j_0: U_0 \hookrightarrow \mathbf{P}^1_{\overline{\mathbf{F}}_q} \). Let \( i_0: Z_0 \hookrightarrow \mathbf{P}^1_{\mathbf{F}_q} \) be the complement of \( U_0 \). Then, we get the short exact sequence

\[
0 \longrightarrow j_{0!} \mathbf{Q}_\ell \longrightarrow j_! \mathbf{Q}_\ell \longrightarrow \mathbf{Q}_\ell \longrightarrow 0.
\]

Note \( j_! \mathbf{Q}_\ell \cong \overline{\mathbf{Q}}_\ell \) since \( (\mathbf{P}^1_{\mathbf{F}_q})^\text{th} - \{ x \} \) is connected, and that \( Q = i_* \overline{\mathbf{Q}}_\ell \). Now since the short exact sequence is completely Galois invariant, we have that the associated long exact sequence on cohomology

\[
0 \longrightarrow \mathbf{Q}_\ell^{(\#Z_0 - 1)} \longrightarrow H_c^1(U, \overline{\mathbf{Q}}_\ell) \longrightarrow H^1(\mathbf{P}^1, \overline{\mathbf{Q}}_\ell) = 0
\]

is exact, where the \(-1\) in the exponent of \( \mathbf{Q}_\ell \) comes from the fact that we get a copy of \( \mathbf{Q}_\ell \) from \( H^0 \) on \( \mathbf{P}^1 \). Since the middle group has weight 0, we are done.

We can also prove this using semicontinuity of weights: without knowing what \( Q \) is in the short exact sequence above, we still know \( Q \) has finite support, and that \( w(Q) \leq w \).

We now state the key assertions that we will use to prove Theorem 8.1.

**Key Assertions 8.2.** Denote \( G_0 = j_{0!}(F_0) \) and denote \( j_0: U_0 \hookrightarrow \mathbf{A}_F^1 \). Let \( \psi: \mathbf{F}_q \rightarrow \overline{\mathbf{Q}}_\ell \) be a fixed non-trivial additive character. Then,

(a) \( T_{\psi}(G_0) \) is a sheaf, that is, the complex \( T_{\psi}(G_0) \) is concentrated in degree 0;

(b) \( H_c^0(\mathbf{A}^1, T_{\psi}(G_0)) = 0 \);

(c) \( T_{\psi}(G_0) \) is \( \tau \)-mixed.

**Remark 8.3.** Condition (a) fails if \( G_0 \) is geometrically constant, which explains why we needed reduction (7).

**Proof of Theorem 8.1 assuming Key Assertions.** Recall that the Fourier transform switches taking stalks and computing cohomology (Theorem 7.13), that is,

\[
(T_{\psi}(G_0))|_{\{0\}} = R\Gamma_c(\mathbf{A}^1, G)[1].
\]

But the right-hand side is \( R\Gamma_c(\mathbf{A}^1, G)[1] = R\Gamma_c(U, F)[1] = H_c^1(U, F)[0] \) by assertion (a).

To understand the Frobenius eigenvalues of \( H_c^1(U, F) \), then, it suffices to understand the Frobenius eigenvalues, hence the weights, of the Fourier transform; in particular, we want to show

\[
w(T_{\psi}(G_0)) \leq w + 1.
\]

We showed (Theorem 4.11) that the weights of \( T_{\psi}(G_0) \) satisfy \( w(T_{\psi}(G_0)) = \|T_{\psi}(G_0)\| \), where we note that in order to apply Theorem 4.11 we needed assertions (b) and (c); the same Theorem also says \( w(G_0) = \|G_0\| \).

Therefore, (16) holds if and only if \( \|T_{\psi}(G_0)\| \leq \|G_0\| + 1 \), which is exactly the Plancherel formula (Theorem 7.15, see also Remark 7.16) from last time. \( \square \)
Proof of Key Assertions (a) and (b). Instead of following [KW01, §I.6], we use the perverse equivalence (Theorem 7.24) from last time. Since (a) and (b) are both geometric statements, we can pass to the algebraic closure, and consider the sheaves $F$ and $G = j_!(F)$, where $F$ is an irreducible local system on $U$. Then, the shift $G[1]$ of $G$ is a simple, irreducible perverse sheaf on $\mathbb{A}^1$, since $j$ is affine. By Theorem 7.24, this is equivalent to saying $T_\psi(G[1])$ is an irreducible perverse sheaf on $\mathbb{A}^1$. But there is a classification of these:

$$T_\psi(G[1]) = \begin{cases} k_!(M[1]) & \text{where } k: V \hookrightarrow \mathbb{A}^1 \text{ is a dense open, and } M \text{ is an irreducible local system on } V; \\
\iota_*N & \text{where } \iota: \{x\} \hookrightarrow \mathbb{A}^1 \text{ and } N \text{ is an irreducible local system on } \{x\}, \text{ i.e., } N = \mathbb{Q}_\ell. \end{cases}$$

We want to show that the second situation doesn’t happen. Note that by the way the perverse $t$-structure works, we have a shift $[1]$ in the first case but not the second.

So suppose for sake of contradiction that $T_\psi(G[1]) = \iota_*\mathbb{Q}_\ell$. We know that $T_\psi(\mathcal{L}(\psi_{-x})) = \iota_*\mathbb{Q}_\ell[-1]$, and so $G$ is isomorphic to the Artin–Schreier sheaf $L(\psi_{-x})$. But these are always ramified at infinity (since otherwise you get a finite étale cover of $\mathbb{P}^1$), unless $x = 0$, in which case we have that $G$ is isomorphic to the constant sheaf $\mathbb{Q}_\ell$, which we’ve already ruled out.

Thus, $T_\psi(G[1]) = k_!(M[1])$ where $k: V \hookrightarrow \mathbb{A}^1$ is a dense open, and $M$ is an irreducible local system on $V$. Canceling the shifts on either side, we have that $T_\psi(G) = k_!(M)$, in which case the assertions are clear: this is clearly a sheaf, and $H^0_c(\mathbb{A}^1, T_\psi(G)) = H^0_c(V, M) = 0$. $\square$

We therefore see that (a) and (b) are really just consequence of the abstract Fourier transform machinery, without having to do anything new and difficult. We now move onto showing the remaining Key Assertion:

Proof of Key Assertion (c). We want to use the last piece of machinery we have yet to use: the fact that summands of $\tau$-real sheaves are $\tau$-mixed. Note that this is really the only way we know of to prove that a sheaf is $\tau$-mixed.

Consider the Fourier transform setup from last time

$$\begin{array}{ccccc}
\mathbb{A}^1_{\overline{F}_q} \times \mathbb{A}^1_{\overline{F}_q} & \xrightarrow{m} & \mathbb{A}^1_{\overline{F}_q} \\
\pi^! & \downarrow & \pi^! \\
\mathbb{A}^1_{\overline{F}_q} & \xleftarrow{} & \mathbb{A}^1_{\overline{F}_q}
\end{array}$$

Recall that $G_0 = j_0!(F_0)$, and so its Fourier transform is $T_\psi(G_0) := R\pi_!^1(\pi^2*G_0 \otimes m^*\mathcal{L}(\psi))[1]$.

Consider the following $\tau$-real sheaf formed by summing the sheaf used in the definition of the Fourier transform (before taking $R\pi_!^1$) with its dual:

$$H_0 = \pi^2*(j_0!(F_0)) \otimes m^*(\mathcal{L}(\psi)) \oplus (\pi^2*(j_0!(F_0)) \otimes m^*(\mathcal{L}(\psi^{-1}))) \otimes \mathcal{L}_b,$$

where $b \in \overline{\mathbb{Q}_\ell}$ is such that $\tau(b) = q^m$. By construction, $H_0$ is $\tau$-real. Now $R^i\pi_!^1(A) = 0$ if $i \neq 1$ by assertion (a). In the same way, you can show $R^i\pi_!^1(B) = 0$ if $i \neq 1$. By the Grothendieck trace formula for $R\pi_!^1(H_0)_\pi$, where $\pi$ is a geometric point living over a closed point $x \in |\mathbb{A}^1|$, since the higher direct images for $i \neq 1$ vanish, we have

$$\det \left( 1 - t \cdot F^d(x) \bigg| (R^1\pi_!^1(H_0))_\pi \right) = \prod_{y \in \pi^{-1}(\pi^1)(x)} \det \left( 1 - t \cdot F^d(y) \bigg| (H_0)_\pi \right)^{-1}$$

The right-hand side has $\mathbb{R}$-coefficients by construction, and so the left-hand side has $\mathbb{R}$-coefficients, that is, $R\pi_!^1(H_0)$ is $\tau$-real. By Theorem 6.9, the summand $R\pi_!^1(A) = T_\psi(G_0)[-1]$ is $\tau$-mixed. $\square$

This concludes the proof of the Weil conjectures in the curve case.
8.2 General Case

Theorem 8.4. Let $X_0$ be a scheme of finite type over $\mathbb{F}_q$, and let $F_0$ be a $\mathbb{Q}_\ell$ sheaf on $X_0$, which is $\tau$-mixed of weight $\leq w$. Then, $H^i_c(X, F)$ is $\tau$-mixed of weight $\leq w + i$.

Corollary 8.5. If $f_0 : X_0 \to Y_0$ is a smooth, proper map of schemes of finite type, and $F_0$ is $\tau$-pure of weight $\beta$, then $R^jf_0(F_0)$ is $\tau$-pure of weight $\beta + i$.

Proof of Corollary 8.5. By Theorem 8.4 apply base change to identify stalks.

Proof of Theorem 8.4. We induce on the dimension $d$ of $X_0$.

1. $d = 0$ is okay: it’s just a finite union of points.

2. We may replace $(X_0, F_0)$ by $(U_0, F_0|\mid U_0)$ just as before, where $U_0 \to X_0$ is a dense open: If $U_0 \to X_0 \leftarrow Z_0 = X_0 \setminus U_0$, then we get the short exact sequence

$$0 \to j_! (F_0|\mid U_0) \to F_0 \to i_!(F_0|\mid Z_0) \to 0$$

and the long exact sequence on cohomology says

$$\cdots \to H^i_c(U, F|\mid U) \to H^i_c(X, F) \to H^i_c(Z, F|\mid Z) \to \cdots$$

is exact. We can control the $H^i_c(Z, F|\mid Z)$ term by induction.

3. We may extend base fields.

4. To finish the proof, first reduce to the case where $X_0$ is a smooth affine variety, and $F_0$ is a local system on $X_0$ by (2). Then, we may assume there exists a map $\pi : X_0 \to Y_0$ is a smooth, affine map of relative dimension 1, e.g., by using noether normalization to reduce to the case where $X_0$ is an (open subset of an) affine space, and then by using a coordinate projection, passing to smaller open subsets as necessary to keep things smooth. We are therefore in a situation where we have $X_0$ fibred over $Y_0$ by curves.

We then have a Leray spectral sequence

$$E^{i,j}_2 : H^i_c(Y, R^j\pi_0(F)) \Rightarrow H^{i+j}_c(X, F).$$

Claim 8.6. $R^j\pi_0(F)$ is $\tau$-mixed of weight $\leq w + j$.

The Claim plus induction on dimension implies $E^{i,j}_2$ is $\tau$-mixed of weights $\leq w + i + j$, and so $E^{i,j}_\infty$ is $\tau$-mixed of weights $\leq w + i + j$. There is a subtlety in showing this Claim, however, and requires one more reduction, which we will incorporate in the proof next time.

Next time we will finish the end of the proof of Theorem 8.4 and discuss Hard Lefschetz.

9 June 8—Weil Conjectures II & Hard Lefschetz (Bhargav Bhatt)

9.1 Weil Conjectures

Last time, we proved:

Theorem 9.1. Let $U_0$ be a smooth affine curve over $\mathbb{F}_q$, and let $F_0$ be a smooth $\mathbb{Q}_\ell$-sheaf, which is $\tau$-mixed of weights $\leq w$. Then, $H^i_c(U, F)$ is $\tau$-mixed of weights $\leq w + i$.

Example 9.2. Let $E_0$ be an elliptic curve over $\mathbb{F}_q$, and choose two rational points $x, y \in E_0(\mathbb{F}_q)$ that are distinct. Let $U_0$ be the complement $E_0 \setminus \{x, y\}$. This is a smooth affine curve. Let $F_0 = \mathbb{Q}_\ell$.

Claim 9.3. $H^i_c(U, \mathbb{Q}_\ell)$ is $\tau$-mixed of weights 0, 1, i.e., some constituents have weight 0 and others have weight 1.
Proof. Let \( U_0 \xrightarrow{j_0} E_0 \xleftarrow{i_0} Z_0 = \{x, y\} \). Then, we have the usual short exact sequence
\[
0 \rightarrow j_0_!(\mathbb{Q}_\ell) \rightarrow \mathbb{Q}_\ell \rightarrow i_0_*(\mathbb{Q}_\ell) \rightarrow 0
\]
Then, the long exact sequence on cohomology says
\[
0 \rightarrow H^0(Z, \mathbb{Q}_\ell) \rightarrow H^0(E, \mathbb{Q}_\ell) \rightarrow H^1_c(U, \mathbb{Q}_\ell) \rightarrow H^1(E, \mathbb{Q}_\ell) \rightarrow 0
\]
Remarks 9.4.
1. If \( F_0 \) is \( \tau \)-pure of weight \( w \) (we reduced to this case anyways), then \( H^1_c(U, F) \) is \( \tau \)-mixed of weights \( \leq w + 1 \), and \( H^2_c(U, F) \) is \( \tau \)-pure of weight \( w + 2 \) (by Poincaré duality, since \( H^0 \) is a stalk). We used this to show there was no cancellation in the \( L \)-function
\[
L(U_0, F_0, t) = \frac{\det(1 - t \cdot F | H^1_c(U, F))}{\det(1 - t \cdot F | H^2_c(U, F))}
\]
and so if \( L(U_0, F_0, t) \) has \( \mathbf{R} \)-coefficients, the same also holds for \( \det(1 - t \cdot F | H^1_c(U, F)) \) for both \( i = 1, 2 \).
2. Fix \( f_0 : X_0 \rightarrow Y_0 \) a smooth affine map of relative dimension 1, and let \( F_0 \) be a \( \tau \)-pure smooth \( \mathbb{Q}_\ell \)-sheaf on \( X_0 \) of weight \( w \). We then want to understand the higher pushforwards of \( F_0 \): \( R^if_0_!(F_0) \) is \( \tau \)-mixed of weight \( \leq w + i \), and is \( \tau \)-pure of weight \( w + 2 \) if \( i = 2 \).
We now prove the general case.

Theorem 9.5. Let \( X_0 \) be a scheme of finite type over \( \mathbf{F}_q \), and let \( F_0 \) be any \( \tau \)-mixed sheaf of weight \( \leq w \). Then, \( H^i_c(X, F) \) is \( \tau \)-mixed of weight \( \leq w + i \).

Corollary 9.6. If \( X_0 \) is a smooth proper variety over \( \mathbf{F}_q \) of dimension \( d \), and \( F_0 \) is \( \tau \)-pure of weight \( w \), then \( H^i(X, F) \) is also \( \tau \)-pure of weight \( w + i \).

Proof of Corollary 9.6. By the Theorem 9.5, we have that \( H^i(X, F) \) is \( \tau \)-mixed of weights \( \leq w + i \). By duality,
\[
H^i(X, F) \cong (H^{2d-i}(X, F^\vee))^\vee(-d).
\]
\( H^{2d-i}(X, F^\vee) \) has weights \( \leq -w + 2d - i \), and so the right-hand side has weights \( \geq w - 2d + i + 2d = w + i \).

Proof of Theorem 9.5. Let \( d = \dim(X_0) \). We work by induction on \( d \). We make the following reductions, as we did last time:
- We may assume \( X_0 \) is a smooth affine variety, and \( F_0 \) is a smooth \( \mathbb{Q}_\ell \)-sheaf.
- We may assume \( F_0 \) is actually \( \tau \)-pure of weight \( w \) (we forgot to say this last time), by using long exact sequences.
- We may assume that there exists \( \pi_0 : X_0 \rightarrow Y_0 \) a smooth affine map of relative dimension 1. [It might be best to do the reductions in the opposite order.] We now have the following Leray spectral sequence:
\[
E_2^{ij} : H^i_c(Y, R^j\pi_0_!(F)) \Rightarrow H^{i+j}_c(X, F)
\]
Remark 9.4.2) to Theorem 9.1 shows that \( R^j\pi_0_!(F) \) is \( \tau \)-mixed of weight \( \leq w + j \). Induction then implies the result: \( E_2^{ij} \) is \( \tau \)-mixed of weights \( \leq w + i + j \), and so \( E_\infty^{ij} \) is \( \tau \)-mixed of weights \( \leq w + j + i \).

Remark 9.7. Using similar arguments, one can show: If \( f_0 : X_0 \rightarrow Y_0 \) is a (separated) map of finite type \( \mathbf{F}_q \)-schemes, and \( F_0 \) is a \( \tau \)-mixed sheaf of weights \( \leq w \) on \( X_0 \), then \( R^if_0_!(F_0) \) is \( \tau \)-mixed of weights \( \leq w + i \). There is also a dual version to Theorem 9.5.

Theorem 9.5. Let \( X_0 \) be a smooth variety over \( \mathbf{F}_q \), and let \( F_0 \) be \( \tau \)-pure of weight \( w \) and lisse. Then, \( H^i(X, F) \) is \( \tau \)-mixed of weights \( \geq w + i \).

Proof. \( H^i(X, F) \cong H^{2d-i}(X, F^\vee)^\vee(-d) \), and conclude as in the proof of Corollary 9.6.
9.2 Hard Lefschetz

We now want to prove Hard Lefschetz, which is a statement about how capping with the Chern class of an ample line bundle acts on cohomology. We first start with a theorem about monodromy representations:

**Theorem 9.8.** Let $X$ be smooth and geometrically connected over $\mathbb{F}_q$. Let $F_0$ be $\tau$-pure of weight $w$ and lisse. Recall that this corresponds to some representation $\rho \in \text{Rep}_{\text{arith}}(\pi_1^{\text{arith}}(X_0))$. Then, $\rho|_{\pi_1^{\text{geom}}(X)}$ is semisimple, that is, “any pure sheaf has semisimple geometric monodromy.”

**Proof.** We work by induction on $l(\rho)$, the length of $\rho$, defined to be the number of semisimple constituents of $\rho$ in $\text{Rep}(\pi_1^{\text{arith}}(X_0))$. If $l(\rho) = 1$, then $\rho$ is irreducible, and since $\pi_1^{\text{geom}}(X) \subset \pi_1^{\text{arith}}(X_0)$ is normal, Clifford’s theorem implies $\rho|_{\pi_1^{\text{geom}}(X)}$ is semisimple.

Now assume $l(\rho) > 1$, and so there exists a short exact sequence

$$0 \rightarrow A_0 \rightarrow F_0 \rightarrow B_0 \rightarrow 0$$

where $A_0, B_0$ are both $\tau$-pure of the same weight $w$ as $F_0$, and $A_0, B_0$ are both nonzero. We know by induction that $A_0, B_0$ both induce semisimple representations. We want to show that (17) is split after removing the zeros. Note that (17) is classified by some element $c \in \text{Ext}^1_{X_0}(B_0, A_0)$. We want that the image of $c$ in $\text{Ext}^1_{X}(B, A)$ is zero. This follows by combining two observations:

1. $\text{im}(c)$ is Frobenius-invariant of $\text{Ext}^1_{X_0}(B, A) = H^1(X, B^\vee \otimes A)$.
2. $B^\vee \otimes A$ is $\tau$-pure of weight 0, and so Theorem 2 implies $H^1(X, B^\vee \otimes A)$ is $\tau$-mixed of weights $\geq 1$. This creates a mismatch of weights: $\text{im}(c)$ cannot simultaneously be Frobenius-invariant and have eigenvalues of weight $\geq 1$.

**Example 9.9.** Say $f : X \rightarrow S$ is a smooth proper map over $\mathbb{F}_q$, and $S$ is smooth. Then, Theorem 9.8 implies $R^i f_* \mathbb{Q}_l \in \text{Rep}(\pi_1(S))$ is semisimple (it is lisse since $f$ is proper smooth, and $\tau$-pure of weight $i$ by the Weil conjectures and the duality argument).

This same statement is true over complex numbers, but we are unaware of which came first.

We now introduce “Hard Lefschetz” and “Not-so-hard Lefschetz.”

**Recall 9.10.** Let $k$ be an algebraically closed field; we will mostly consider the case when $k = \mathbb{F}_q$. Let $X$ be a smooth projective variety over $k$ of dimension $d$, and $L \in \text{Pic}(X)$ an ample line bundle. Then, we have the first Chern class $c_1(L) \in H^2(X, \mathbb{Q}_l(1))$ of $L$.

**Theorem 9.11.**

1. [Weak Lefschetz] If $H \subset X$ is a smooth divisor in the linear system $|L|$, then the restriction map

$$H^i(X, \mathbb{Q}_l) \xrightarrow{\text{Res}} H^i(H, \mathbb{Q}_l)$$

is bijective if $i < d - 1$, and injective if $i = d - 1$.

2. [Hard Lefschetz] For each $0 \leq i \leq d$, the map

$$H^{d-i}(X, \mathbb{Q}_l) \xrightarrow{c_1(L)^i} H^{d+i}(X, \mathbb{Q}_l)(i)$$

is bijective.

**Corollary 9.12.** Say $d$ is even, and let $b_i = \dim H^i(X, \mathbb{Q}_l)$. Then, $b_0 \leq b_2 \leq b_4 \leq \cdots \leq b_d$. Likewise, $b_1 \leq b_3 \leq b_5 \leq \cdots \leq b_{d-1}$. Also, $b_i = b_{2d-i}$. (There is a similar statement for $d$ odd.)

We want to show the Lefschetz theorems using the monodromy results we’ve proved. There is a nice theory of Lefschetz pencils, of which we will use a tiny portion.

**Fact 9.13.** After possibly replacing $L$ with $L^\otimes m$, there exist two general elements of $|L|$ that intersect in a codimension 2 set $\Delta \subset X$. Then, we can blow up $\Delta$ to obtain a pencil $\mathcal{Y}$:

$$\begin{align*}
Y_s & \hookrightarrow Y & \subset & \mathcal{Y} & \xrightarrow{f} & X \\
\downarrow & & & \downarrow \pi_{1\nu} & & \downarrow s \\
\{s\} & \hookrightarrow U & \subset & \text{open} & \mathbb{P}^1
\end{align*}$$
where $f$ is the blow up of $\Delta \subset X$, hence proper birational; $\pi$ is a proper map, with all fibres of dimension $d-1$ with at worst a single ordinary double point; and $\pi|_U$ is proper and smooth of relative dimension $d-1$.

In this setup, the cohomology of $X$ and fibres of $Y$ are nicely related: if $s \in U$ is a geometric point, then for $i \leq d-1$,

$$H^i(X, \mathbb{Q}_\ell) \sim (H^i(Y_s, \mathbb{Q}_\ell))^{\pi_1(U)}.$$ 

This is saying that in the range where Weak Lefschetz only gives an injection, we have a bijection after passing to invariants.

**Proof of Hard Lefschetz.** We work by induction on the dimension $d$; the $d = 0$ case is stupid.

Choose a Lefschetz pencil:

$$
\begin{array}{cccccc}
Y_s & \longrightarrow & Y & \longrightarrow & Y & \longrightarrow & X \\
\downarrow & & \downarrow \pi|_U & & \downarrow \pi & & \\
\{s\} & \longrightarrow & U & \longrightarrow & \mathbb{P}^1 & & \\
\end{array}
$$

Set $h: Y_s \hookrightarrow X$, so $h^*: H^i(X) \to H^i(Y_s)$. Then, Weak Lefschetz says

$$h^* \text{ is an } \begin{cases} \text{isomorphism} & \text{if } i < d-1 \\ \text{injection} & \text{if } i = d-1 \end{cases}$$

Duality says

$$h_*: H^j(Y_s) \to H^{j+2}(X) \text{ is an}\begin{cases} \text{isomorphism} & \text{if } j \geq d \\ \text{surjection} & \text{if } j = d-1 \end{cases}$$

Now consider

$$
\begin{align*}
H^{d-1}(X) &\xrightarrow{c_1(L)^{-1}} H^{d+1}(X) & \xrightarrow{c_1(L)} H^{d+1}(X) \\
H^{d-1}(Y_s) &\xrightarrow{c_1(L)^{-1}} H^{d+2}(Y_s) \\
\end{align*}
$$

The diagram commutes since $h_*h^*(\alpha) = \alpha \cdot h_*h^*(1) = \alpha \cdot h_*(1) = \alpha \cdot c_1(L)$. Now if $i \geq 2$, then $h^*, h_*$ are isomorphisms, and so by the diagram $c_1(L)^i$ is an isomorphism by induction.

Now assume $i = 1$. Then, $h^*$ is injective, and $h_*$ is surjective. Using Poincaré duality, we have $H^{d-1}(X) \xrightarrow{c_1(L)^{-1}} H^{d+1}(X)$ is an isomorphism, if and only if the pairing

$$H^{d-1}(X) \times H^{d-1}(X) \to H^{2d}(X) \cong \mathbb{Q}_\ell$$

is non-degenerate. To prove this, using $h^*$, it suffices to show the standard pairing

$$H^{d-1}(Y_s) \times H^{d-1}(Y_s) \to H^{2d-2}(Y_s) \cong \mathbb{Q}_\ell$$

is non-degenerate on $h^*(H^{d-1}(X))$, since we have the commutative diagram

$$
\begin{align*}
(a, b) &\quad \longrightarrow & a \cup c_1(L)b \\
H^{d-1}(X) \times H^{d-1}(X) &\xrightarrow{h^*} H^{2d}(X) \\
H^{d-1}(Y_s) \times H^{d-1}(Y_s) &\xrightarrow{h^*} H^{2d-2}(Y_s) \\
\end{align*}
$$

We observe:

- All objects have an action of $\pi_1(U)$;
- $H^{d-1}(Y_s) \cong H^{d-1}(X)$;
- Theorem 9.8 implies $H^{d-1}(Y_s) \cong H^{d-1}(Y_s) \oplus Q$, where $Q^{\pi_1(U)} = 0$.

We therefore have that the pairing (18) is nondegenerate. 

$$\square$$
References
