Takumi Murayama
Research Statement

Overview. My research is in algebraic geometry. The goal of my research is to investigate the structure theory of algebraic varieties. Specifically, I apply techniques from positive characteristic commutative algebra to extend existing results from the minimal model program and moduli theory to positive characteristic, as well as obtain new results in characteristic zero via reduction modulo \( p \). My research falls under three topics:

1. Global generation of line bundles and Fujita’s conjecture.
2. Matsusaka-type theorems and weak positivity of direct image sheaves.
3. Singularity theory and asymptotic invariants over imperfect fields.

Progress in these areas would further the minimal model program and moduli theory in both positive characteristic and characteristic zero. In the direction of each topic listed above, I proved the following:

1. A criterion for existence of global sections of line bundles of the form \( \omega_X \otimes L^\ell \) with prescribed Taylor series expansions at a point, which I use to characterize when a variety is projective space.
2. A criterion for existence of global sections of sheaves of the form \( f_* \omega_X^{m} \otimes L^\ell \), where \( f: X \to Y \) is a morphism of complex varieties, and an analogue in positive characteristic.
3. A systematic method to work with imperfect fields in positive characteristic, which I use to characterize when a line bundle is ample using asymptotic invariants.

0. Background: The structure theory of algebraic varieties

Let \( k \) be a field. A projective variety over \( k \) is a subset of a projective \( n \)-space \( \mathbf{P}_k^n \) defined by the vanishing locus of a set of homogeneous polynomials. Beginning with Riemann, algebraic geometers have studied how projective varieties can be classified into classes that share common features; see [Kol14]. One approach to this classification problem is to consider the following question:

**Question 0.1.** Given a projective variety \( X \), is there a “simplest” variety \( X^m \) birational to \( X \), i.e., such that \( X \) and \( X^m \) have the same meromorphic functions?

This is the main goal of the minimal model program, which seeks to find a systematic way to find the “simplest” variety \( X^m \). For example, the hyperboloid on the right is defined by the equation \( x^2 + y^2 - z^2 = w^2 \) in \( \mathbf{P}_k^3 \), and can be constructed by sweeping a line through space. While two lines on a hyperboloid may not intersect, two lines always intersect in the projective 2-space \( \mathbf{P}_k^2 \). Nonetheless, hyperboloids are birational to \( \mathbf{P}_k^2 \), since the blowup of a hyperboloid at one point is isomorphic to the blowup of \( \mathbf{P}_k^2 \) at two points. The minimal model program arose from the study of elliptic integrals by Euler, Abel, Jacobi, Riemann, and others, and recently has been applied to the study of rational points in arithmetic geometry, specifically to Manin’s conjecture [LT].

For smooth projective curves \( X \), Question 0.1 is easily answered: \( X \) is birational to another smooth projective curve \( Y \) if and only if \( X \) and \( Y \) are isomorphic. We therefore also ask:

**Question 0.2.** Given a class of varieties with similar invariants, does there exist an algebraic variety parametrizing them?

An algebraic variety parametrizing a class of varieties is an example of a moduli space. Problems in moduli theory ask whether moduli spaces exist for a given class of varieties. For example, one can ask whether curves with a fixed genus can be parametrized by a moduli space. Historically, Question 0.2 was first considered for elliptic curves, where it was shown that two elliptic curves are isomorphic if and only they have the same \( j \)-invariant, in which case the moduli space is \( \mathbf{P}_k^1 \).

One way to understand the difference between the two questions is that the minimal model program seeks to relate varieties via various surgery operations, whereas moduli theory asks whether varieties can be deformed or degenerated to other varieties in algebraic families.

For curves, Teichmüller and Mumford proved that moduli spaces that parametrize all curves of a fixed genus \( g \) exist [HM98]. The structure theory of surfaces is relatively well-understood as well, due to the Enriques–Kodaira classification and its extension to positive characteristic by Mumford and Bombieri [Băd01].
In higher dimensions, the first major breakthrough in the minimal model program was due to Mori [Mor82], who showed that if $X$ is a smooth complex projective threefold, then one can find copies of $\mathbb{P}_k^1$ inside of $X$ that can be contracted to form a simpler space $X'$. This theorem was generalized to higher-dimensional complex projective varieties with mild singularities by Kawamata [Kaw84b] and Kollár [Kol84]. The idea of the minimal model program is then to iterate this process of contracting $\mathbb{P}_k^1$'s; see [KM98]. During this process, one may introduce singularities even if the initial variety was smooth. It is unknown whether this process terminates in general, although it does for surfaces and threefolds [Mor88]. Recent breakthroughs due to Birkar, Cascini, Hacon, and McKernan [BCHM10, HM10] established that the minimal model program works for varieties of general type, which is the class of varieties such that the first Chern class $c_1(X)$ of the tangent bundle is “as negative as possible.”

Solving problems in moduli theory depends highly on what class of varieties one is interested in [FM13]. For many of these moduli problems, an important first step is to find an integer $n$ such that every projective variety with the same Hilbert polynomial has an embedding into $\mathbb{P}_k^n$. The existence of such an $n$ was proved for smooth complex projective varieties by Matsusaka [Mat72]. This result was used by Kollár [Kol90] and Viehweg (see [Vie95]) to show that a quasi-projective moduli space exists for smooth complex varieties with a fixed Hilbert polynomial.

One interesting aspect of Mori’s groundbreaking work is that while his results are for complex varieties, he uses reduction modulo $p$ and positive characteristic methods to prove the existence of rational curves that can be contracted [Mor82], a result which does not have a direct proof over the complex numbers. Nonetheless, in positive characteristic, the structure theory of varieties in higher dimension is less well-developed. The most problematic situation is when the ground field $k$ is an imperfect field of characteristic $p > 0$, in which case there are three major difficulties. First, since $k$ is of characteristic $p > 0$,

(I) Resolutions of singularities are not known to exist (see [Hau10]), and
(II) Vanishing theorems are false [Ray78].

A common workaround for the lack of resolutions is to use de Jong’s theory of alterations [dJ96]. The lack of vanishing theorems is harder to circumvent, since over the complex numbers, vanishing theorems are a fundamental ingredient used to construct global sections of line bundles. A useful workaround is to exploit the Frobenius morphism $F: X \to X$ and its Grothendieck trace $F_!\omega_X^* \to \omega_X^*$; see [PST17]. For imperfect fields, however, this approach runs into another problem:

(III) Most applications of Frobenius techniques require $k$ to be $F$-finite, i.e., satisfy $[k : k^p] < \infty$.

The last issue arises since Grothendieck duality cannot be applied to the Frobenius if it is not finite.

My research aims to develop the structure theory of algebraic varieties in positive characteristic by using commutative-algebraic methods involving the Frobenius morphism to replace vanishing theorems. Since resolutions of singularities are unavailable in positive characteristic, this approach requires a careful study of the singularities involved, especially when the ground field is not $F$-finite. While my methods use the fact that the characteristic is nonzero, my methods produce new results in characteristic zero via reduction modulo $p$.

1. Global generation of line bundles and Fujita’s conjecture

In the minimal model program, the fundamental method used to construct maps that contract $\mathbb{P}_k^1$’s is the cohomological method of Kawamata–Reid–Shokurov [Kaw84a, Rei83, Sho86]. Their method produces global sections of line bundles inductively by using vanishing theorems to lift sections from subvarieties called log canonical centers. In positive characteristic, a fundamental obstruction to this approach is that not only are vanishing theorems not valid, but it is also difficult to construct the positive characteristic analogue of log canonical centers, called centers of $F$-purity [Sch10].

1.1. Past work. My approach for producing global sections is to use Seshadri constants. Let $L$ be a line bundle on a projective variety $X$. We say that $L$ is nef if $\int_C c_1(L) \geq 0$ for every curve $C \subseteq X$. Demailly introduced Seshadri constants to measure the local positivity of a nef line bundle $L$ at a closed point $x \in X$. If $\mu: \bar{X} \to X$ is the blowup of $X$ at $x$ with exceptional divisor $E$, then the Seshadri constant of $L$ at $x$ is

$$\varepsilon(L; x) := \sup\{t \in \mathbb{R}_{\geq 0} \mid \mu^* L(-tE) \text{ is nef}\}.$$
A fundamental property of Seshadri constants is that if \( \varepsilon(L; x) \) is sufficiently large, then \( \omega_X \otimes L \) has global sections with prescribed Taylor series expansions at \( X \), as seen in the following result due to Demailly for smooth complex varieties [Dem92], and myself in general [Mur18]. A special case of the positive characteristic result is due to Mustaţă and Schwede [MS14].

**Theorem 1.1** (Murayama). Let \( X \) be a projective variety of dimension \( n \), and let \( L \) be a nef line bundle on \( X \) such that \( \varepsilon(L; x) > n + s \) for some integer \( s \geq 0 \). If \( X \) has rational singularities at \( x \) in characteristic zero, or \( F \)-injective singularities at \( x \) in positive characteristic, then the following restriction morphism is surjective:

\[
H^0(X, \omega_X \otimes L) \rightarrow H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X/m_x^{s+1}).
\]

Here, \( F \)-injective singularities are the positive-characteristic analogue of Du Bois singularities in characteristic zero [Sch09]. As an application, I proved the following characterization of projective space [Mur18] that is analogous to a result of Bauer and Szemberg over the complex numbers [BS09]:

**Theorem 1.2** (Murayama). Let \( X \) be a smooth Fano variety of dimension \( n \) over an algebraically closed field of positive characteristic. If there exists a closed point \( x \in X \) such that \( \int_C c_1(\omega_X^{-1}) \geq (\text{mult}_x C) \cdot (n+1) \) for every integral curve \( C \subseteq X \) passing through \( x \), then \( X \) is isomorphic to \( \mathbb{P}^n \).

To use Theorem 1.1 to produce global sections of line bundles, one needs to ensure a lower bound of the form \( \varepsilon(L; x) > n + s \) holds for some point \( x \in X \), where \( n = \dim X \). Ein, Lazarsfeld, and Küchle [EL93b, EKL95] showed that over the complex numbers, \( \varepsilon(L; x) \geq 1/n \) at very general points \( x \in X \), hence Theorem 1.1 can be applied to \( L^{\otimes (n+s)+1} \). However, such lower bounds that depend solely on the dimension are unknown in positive characteristic. My approach is to replace the dimension \( n \) in the bound \( \varepsilon(L; x) > n + s \) with other invariants involving the singularities of a pair \((X, \Delta)\) in the sense of [Kol96], where \( \Delta \) is a \( \mathbb{Q} \)-linear combination of codimension one subvarieties in \( X \). Currently, I am able to prove the following result using Frobenius techniques and the Frobenius degeneracy ideals \( I^p_\Delta \) introduced by Aberbach–Enescu [AE05] and Blickle–Schwede–Tucker [BST12]:

**Theorem 1.3** (Murayama). Let \( X \) be a normal projective variety over a field \( k \) of characteristic \( p > 0 \) such that \( [k : k^p] < \infty \), and let \( \Delta \) be an effective \( \mathbb{Q} \)-Weil divisor on \( X \) such that \((p^e - 1)(K_X + \Delta) \) is Cartier for some \( e > 0 \). Consider a closed point \( x \in X \) such that \((X, \Delta)\) is strongly \( F \)-regular at \( x \). Let \( D \) be a Cartier divisor on \( X \), and suppose \( H = D - (K_X + \Delta) \) is nef and \( \varepsilon(H; x) > \text{fpt}_x((X, \Delta); m_x) \). Then, \( \mathcal{O}_X(D) \) is globally generated at \( x \).

Strong \( F \)-regularity is the positive characteristic analogue of Kawamata log terminal singularities, and the \( F \)-pure threshold, \( \text{fpt} \), is the positive characteristic analogue of the log canonical threshold, let [Smi00, Har01, HY03, Tak04]. This analogy is strong enough that by reduction modulo \( p \), one obtains a version of Theorem 1.3 with “strongly \( F \)-regular” replaced by “Kawamata log terminal” and with “\( \text{fpt} \)” replaced by “\( \text{ft} \)”.

1.2. **Future directions.** Demailly originally defined Seshadri constants to study Fujita’s conjecture [Fuj88], which states the following:

**Conjecture 1.4** (Fujita). Let \( X \) be a smooth projective variety with canonical bundle \( \omega_X \), and let \( L \) be an ample line bundle on \( X \). Then, \( \omega_X \otimes L^\ell \) is globally generated for \( \ell \geq \dim X + 1 \).

In characteristic zero, Fujita’s conjecture is known in dimensions \( \leq 5 \) [Rei88, EL93a, Kaw97, YZ], and a bound for \( \ell \) quadratic in \( \dim X \) is known to suffice [AS95]. The main difficulty in these results is to construct certain singular divisors with isolated log canonical centers, after which one can apply Nadel vanishing to produce global sections. Given the existence of such divisors, Theorem 1.3 can be used instead of Nadel vanishing to reprove these results in characteristic zero. On the other hand, Fujita’s conjecture is unknown in positive characteristic, even for surfaces, unless \( L \) is also assumed to be globally generated [Smi97]. I am currently working toward proving Fujita’s conjecture for surfaces in positive characteristic by producing divisors \( \Delta \) with a small \( F \)-pure threshold at a given point \( x \). This would strengthen existing results due to Shepherd-Barron [SB91], Terakawa [Ter99], and Di Cerbo–Fanelli [DCF15]. My ultimate goal is to investigate to what extent Theorem 1.3 can be used to generalize [AS95] to positive characteristic, and whether it can be used to replace the cohomological method of Kawamata–Reid–Shokurov in the minimal model program.
2. Matsusaka-type theorems and weak positivity of direct image sheaves

As mentioned in §0, Kollár [Kol90] and Viehweg [Vie95] proved that there exists a quasi-projective moduli space parametrizing all polarized smooth complex projective varieties with a fixed Hilbert polynomial. Two important elements involved in proving their theorem are the following:

1. (Boundedness) Prove that all polarized smooth projective varieties $X$ in a certain class and with a fixed Hilbert polynomial can be embedded into a fixed projective space;
2. ((Quasi-)projectivity) Prove that the subscheme of the Hilbert scheme parametrizing that class is (quasi-)projective.

In characteristic zero, Matsusaka’s big theorem [Mat72] asserts that the moduli of smooth complex projective varieties with a fixed Hilbert polynomial is bounded. A fundamental issue in positive characteristic is that Matsusaka’s big theorem [Mat72] is unknown in dimensions $\geq 3$ (the surface case is due to Matsusaka and Mumford [MM64]). Moreover, to prove quasi-projectivity, one needs to prove analogues of Viehweg’s weak positivity theorem (see [Vie95]), which are only known in some special cases in positive characteristic.

2.1. Past work. Over the complex numbers, Popa and Schnell [PS14] realized that one way to prove Viehweg’s weak positivity theorem is to establish special cases of the following relative version of Fujita’s conjecture:

**Conjecture 2.1** (Popa–Schnell). Let $f : Y \to X$ be a morphism of smooth complex projective varieties, and let $L$ be an ample line bundle on $X$. For every $m \geq 1$, the sheaf $f_* \omega_Y^\otimes m \otimes L^\otimes \ell$ is globally generated for all $\ell \geq m(\dim X + 1)$.

Popa and Schnell established a version of their conjecture when $L$ is ample and globally generated, and used it to deduce Viehweg’s weak positivity theorem. In joint work with Yajnaseni Dutta [DM], we were able to prove the following result toward Popa and Schnell’s conjecture:

**Theorem 2.2** (Dutta–Murayama). Let $f : Y \to X$ be a fibration of complex projective varieties where $X$ is smooth of dimension $n$. Let $(Y, \Delta)$ be a log canonical pair, and let $L$ be an ample line bundle on $X$. Then, the sheaf $f_* \mathcal{O}_Y(m(K_Y + \Delta)) \otimes L^\otimes \ell$ is globally generated on a dense open set $U$ for every $\ell \geq m(n + 1) + n^2 - n$.

Theorems asserting global generation on some open set were first obtained by Dutta [Dut], and results similar to Theorem 2.2 were obtained by Deng [Den] and Iwai [Iwa] using analytic methods. However, only our result applies to log canonical pairs. Importantly, we proved a new version of Viehweg’s weak positivity theorem for log canonical pairs, extending results of Campana [Cam04] and Fujino [Fuj17].

2.2. Future directions. Currently, progress on Popa and Schnell’s conjecture has been restricted to complex varieties. In positive characteristic, progress has not been made even in the special case where $L$ is also globally generated. In fact, examples of Moret-Bailly [MB81] (see [SZ]) show that as written, Popa and Schnell’s conjecture is false in positive characteristic because of the existence of families of supersingular abelian varieties. However, I was able to following partial result:

**Theorem 2.3** (Murayama). Let $f : Y \to X$ be a separable fibration of smooth projective varieties of characteristic $p > 0$. Suppose the Hasse–Witt matrix of the geometric generic fiber of $f$ is not nilpotent. Let $L$ be an ample and globally generated line bundle on $X$. Then, for every $m \geq 1$, there is a nonzero subsheaf of $f_* (\omega_Y^\otimes m) \otimes L^\otimes \ell$ that is globally generated for all $\ell \geq m(\dim X + 1)$.

This result relies on the theory of Cartier modules introduced by Blickle and Böckle [BB11]. In future work, I plan to use Viehweg’s fiber product trick and Popa and Schnell’s method in [PS14] to describe the nonzero subsheaf $f_* (\omega_Y^\otimes m) \otimes L^\otimes \ell$; when $m = 1$, for example, the sheaf in question is $S^0 f_*(\sigma(X) \otimes \omega_Y) \otimes L^\otimes \ell$, where $S^0 f_* (\sigma(X) \otimes -)$ is the relative version of Schwede’s canonical linear system [Sch14] defined by Hacon and Xu [HX15]. Such a description would be useful to prove generalizations of existing weak positivity results in positive characteristic due to Patakfalvi [Pat14, Pat18] and Ejiri [Eji18].

I am also working toward proving a version of Matsusaka’s big theorem [Mat72] in positive characteristic. So far, Matsusaka and Mumford [MM64] have proved the surface case over algebraically closed fields of characteristic $p > 0$, which has been made effective by Fernández del Busto in characteristic zero [Fd96], and by Ballico [Bal96] and Di Cerbo–Fanelli [DCF15] in characteristic $p > 0$. While some progress was made by Matsusaka for threefolds in positive characteristic [Mat79, Mat81, Mat82], a fundamental issue is that even
for threefolds, there is no way to effectively produce global sections of line bundles of the form $\omega_X \otimes L$, where $L$ is ample. The methods described in §1 would be applicable for this problem. I also have a combinatorial approach to Matsusaka’s theorem via Gröbner degenerations in non-standard graded polynomial rings.

3. Singularity theory and asymptotic invariants over imperfect fields

A technical issue in positive characteristic algebraic geometry is that one often needs the ground field $k$ to be $F$-finite, i.e., satisfy $[k : k^p] < \infty$, to apply Frobenius techniques. This issue arises since Grothendieck duality cannot be applied to the Frobenius morphism if the Frobenius morphism is not finite. Recent advances in the minimal model program over imperfect fields due to Tanaka [Tan18, Tan] suggest that it would be worthwhile to develop a systematic way around this issue.

3.1. Past work. One way to force the ground field $k$ to be $F$-finite is to extend scalars to the algebraic closure or to the perfect closure of $k$. However, these operations change the singularities of $X$ drastically; for example, the regular variety $\{ sx^p + ty^p = 1 \}$ in $A^2_{F_p(s,t)}$ becomes everywhere non-reduced after passing to the perfect closure. To preserve singularities, I proved the following result in [Mur].

**Theorem 3.1** (Murayama). Let $X$ be a scheme essentially of finite type over a field $k$ of characteristic $p > 0$. Then, there exists a purely inseparable field extension $k \subseteq k^p$ such that $[k^p : (k^p)^p] < \infty$, and such that $X \times_k k^p$ has the same singularities as $X$.

This result applies to the following classes of singularities:

(i) local complete intersection, Gorenstein, Cohen–Macaulay, $S_n$;

(ii) regular, $R_n$, normal, weakly normal, reduced;

(iii) strongly $F$-regular, $F$-pure, $F$-rational, $F$-injective;

(iv) klt, log canonical (in dimensions $\leq 3$, and also for pairs).

The classes of singularities in (iii) are defined using the Frobenius morphism $F: X \to X$; see [TW18]. Theorem 3.1 is already being used in the minimal model program for threefolds [DW], and can be used to reprove some results in [Tan18, Tan]. Parts of Theorem 3.1 are due to Hochster and Huneke [HH94] in the affine setting. As an application, I extended to arbitrary ground fields a result proved by de Fernex, Küronya, and Lazarsfeld [dFKL07] over the complex numbers, characterizing ample divisors $L$ by the fact that small perturbations of $L$ have submaximal growth of higher cohomology groups.

**Theorem 3.2** (Murayama). Let $X$ be a projective variety over an arbitrary field. An $\mathbb{R}$-Cartier divisor $L$ on $X$ is ample if and only if there exists a very ample Cartier divisor $A$ and $\varepsilon \in \mathbb{R}_{>0}$ such that

$$\hat{h}^i(X, L - tA) := \limsup_{m \to \infty} \frac{h^i(X, \mathcal{O}_X([m(L-tA)]))}{m^{\dim X} ((\dim X)!)} = 0$$

for all $i > 0$ and all $t \in [0, \varepsilon)$.

Here, the $\hat{h}^i(X, -)$ are the asymptotic higher cohomological functions introduced by Küronya [Kühr06]. This result is being used to study the behavior of $F$-signature under birational morphisms [MPST].

3.2. Future directions. In joint work with Rankeya Datta, I am using Theorem 3.1 to study classes of singularities in characteristic $p > 0$. For rings $R$ such that the Frobenius $F: R \to R$ is finite, there is a nice theory of singularities of $R$ based on different properties of the Frobenius. For example, $R$ is $F$-pure if the Frobenius is universally injective, and is $F$-split if the Frobenius splits as a map of $R$-modules. We have been able to show that $F$-purity and $F$-splitting coincide for many rings by using versions of Theorem 3.1, and our goal is to show that these notions coincide for all excellent rings. This equivalence is not known even for excellent discrete valuation rings.

I am using Theorem 3.2 to study Seshadri constants on singular varieties in joint work with Mihai Fulger [FM]. Seshadri constants have many equivalent definitions for smooth complex varieties, and we have been able to show that these definitions are still equivalent for singular varieties over arbitrary fields. We are currently working on extending these results to Seshadri constants for vector bundles, in order to give concrete geometric applications to cones of divisors on products of curves, and to prove a version of Theorem 1.2 where the condition is put on the tangent bundle $T_X$ instead of on $\omega_X^{-1}$.


