APPLICATIONS OF LOCAL COHOMOLOGY

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Abstract. Local cohomology was discovered in the 1960s as a tool to study sheaves and their cohomology in algebraic geometry, but have since seen wide use in commutative algebra. An example of their use is to answer the question: how many elements are necessary to generate a given ideal, up to radical?

For example, consider two planes in 4-space meeting at a point. The vanishing ideal \( I = (x, y) \cap (u, v) \subseteq k[x, y, u, v] \) can be generated up to radical by \( xu, yv, xv + yu \). Krull’s Hauptidealsatz implies that one element is not enough, but local cohomology is used to show two elements also do not work.

The main sources for this talk are [Hum07] and [Eis05, App. 1]. For a more “homological” introduction, see [Wei94, §4.6]. The speaker would like to point out that the algebro-geometric literature on the topic takes a different approach via sheaf cohomology; see [Har67; Har77].

1. Introduction: Complete intersections

We start off with our favorite counterexample from Algebraic Geometry I.

Example 1 (Twisted Cubic [Har77, Exc. I.2.17]). Let \( C \) be the twisted cubic curve, defined by the equations \( V(x^2 - yu, xz - y^2, xy - zw) \subseteq \mathbb{P}^3 \).

In algebraic geometry, this is usually the first example of a variety that is not a complete intersection, that is, its vanishing ideal \( I(C) \) cannot be generated by \( r \) elements, where \( r \) is the codimension of \( C \). You can see this by looking at the degree 2 piece of \( I(C) \), which is three-dimensional. However, the vanishing ideal of \( I(C) \) is generated up to radical by \( x^2 - yw, z(xy - zw) + y(xz - y^2) \),

since
\[
(xz - y^2)^2 = x^2z^2 - 2xy^2z + y^4 = z^2(x^2 - yw) - y(z(xy - zw) + y(xz - y^2))
\]
\[
(xy - zw)^2 = x^2y^2 - 2xzw + z^2w^2 = y^2(x^2 - yw) - w(z(xy - zw) + y(xz - y^2))
\]

We can then naively ask: can we generate \( I(C) \), up to radical, by one element? This is impossible by Krull’s Hauptidealsatz, since then \( C \) would have dimension 2, not 1. We say that \( C \) is a set-theoretic complete intersection, that is, there are two hypersurfaces in \( \mathbb{P}^3 \) that cut out \( C \) set-theoretically.

Even in one dimension higher, it’s already hard to answer such questions.

Example 2 (Hartshorne’s Example [Har70, Exc. III.5.12; Har77, Exc. III.4.9]). Let \( Y \subseteq \mathbb{A}^2 \) be the union of two planes meeting at a point (you can also think of this example as two skew lines in \( \mathbb{P}^3 \)). Its vanishing ideal is
\[ I(Y) = (x, y) \cap (u, v) = (xu, xv, yu, yv) \subseteq k[x, y, u, v], \]
which is generated up to radical by the elements \( xu, yv, xv + yu \), since
\[
(xv)^2 = xv(xv + yu) - (xu)(yu) \in (xu, yv, xv + yu).
\]

We can ask again: can we generate \( I(Y) \), up to radical, by less than three elements? We know it cannot be generated by one element by Krull’s Hauptidealsatz again, since then \( Y \) would have dimension 3. But we can’t rule out two elements by dimension arguments alone!

We want to answer the question: is \( Y \) a set-theoretic complete intersection? To do so, we want to construct some sort of obstruction for \( I(Y) \) to be generated by two elements, up to radical. To a ring \( R \) and an ideal \( J \), we will construct a sequence of modules \( H^j_J(R) \) such that

Properties 3.

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(1) $H^j_I(R) = H^j_{\sqrt{I}}(R)$, and
(2) if $J$ is generated by $k$ elements, then $H^j_J(R) = 0$ for all $i > k$.

These modules are what we will call local cohomology modules.

2. LOCAL COHOMOLOGY AS A DERIVED FUNCTOR

Just like sheaf cohomology, there are multiple ways to define local cohomology. We will describe the derived functor construction first.

Let $R$ be a noetherian ring, $I \subseteq R$ an ideal, and $M$ an $R$-module. We define the $I$-power torsion module of $M$ to be

$$H^0_I(M) := \{ m \in M \mid I^d m = 0 \text{ for some } d \in \mathbb{N} \} = \varinjlim \text{Hom}(R/I^d, M).$$

Note that $\text{Hom}(R/I^d, -)$ is left exact and $\varinjlim$ is exact, and so $H^0_I$ is a left exact functor. Like any left exact functor, we can define its right derived functors, by choosing an injective resolution

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots,$$

applying the functor $H^0_I(\cdot)$ term by term to get a new complex $H^0_I(E^\bullet)$, and then computing cohomology to get modules $H^i_I(M)$. We automatically get some really nice properties!

Properties 4.

(1) If $I$ and $J$ have the same radical, they define the same functor $H^j_I(-)$ (Property 1 from before).
(2) If $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ is a short exact sequence of $R$-modules, there is a long exact sequence

$$0 \rightarrow H^0_I(N) \rightarrow H^0_I(M) \rightarrow H^0_I(L) \rightarrow H^1_I(N) \rightarrow H^1_I(M) \rightarrow \cdots.$$

(3) Every element of $H^i_I(M)$ is killed by some power of $I$.
(4) Since $\varinjlim$ is exact, we can compute local cohomology as

$$H^i_I(M) \cong \varinjlim \text{Ext}^i(R/I^d, M).$$

Before we get too carried away with technical details, let’s compute some examples.

Example 5. Let $p \in \mathbb{Z}$ be a prime number. We will compute $H^1_I(\mathbb{Z})$, where $I = (p)$. Since $\mathbb{Z}$ is a PID, the injective modules are exactly the divisible modules. We have the following injective resolution of $\mathbb{Z}$:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

Now since $H^0_I$ simply computes $p^d$-torsion for all $d$, the only non-vanishing term after applying $H^1_I(\cdot)$ is $H^0_I(\mathbb{Q}/\mathbb{Z})$, and so we have

$$H^1_I(\mathbb{Z}) = H^0_I(\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}[p^{-1}]/\mathbb{Z},$$

where the last isomorphism is by unique factorization.

Similarly, we can compute the local cohomology of $R = k[x]$, where $k$ is a field:

Exercise 6 ([Hum07, Ex. 2.12]). Let $R = k[x]$ be a polynomial ring in one variable over a field $k$. Our goal is to completely describe $H^1_I(M)$ for $I = (x)$ and $M$ an arbitrary finitely generated module.

(1) Using the structure theorem for finitely generated modules over a PID, reduce to the case of computing $H^1_I(R/(g))$ for $g \in R$.

Proof. The structure theorem says

$$M = \bigoplus_{j=1}^s R/(g_j)$$

for some elements $g_j \in R$, possibly equal to zero. We conclude by the fact that computing local cohomology commutes with direct sums. □

(2) First compute the case when $g = 0$ by considering the short exact sequence

$$0 \rightarrow R \rightarrow K \rightarrow K/R \rightarrow 0,$$

where $K = k(x)$ is the fraction field of $k[x].$
Then, there is a long exact sequence in local cohomology

\[ \text{Theorem 7.} \]

For any \( I \) be ideals in a noetherian ring \( R \), and let \( M \) be a finitely generated module \( M \). Then, there is a long exact sequence in local cohomology

\[ 0 \rightarrow H^i_I(M) \rightarrow H^i_I(M) \oplus H^0_J(M) \rightarrow H^0_{I,I}(M) \rightarrow H^0_{I,I}(M) \oplus H^1_J(M) \rightarrow \cdots \]

Proof. We will use the identification \( H^1_J(M) \cong \varinjlim \text{Ext}^1(R/I^n, M) \) from before. Take the short exact sequences

\[ 0 \rightarrow R/(I^n \cap J^n) \rightarrow R/I^n \oplus R/J^n \rightarrow R/(I^n + J^n) \rightarrow 0 \]

for each \( n \), and consider their corresponding long exact sequences on Ext modules. Then, since the system \( \{I^n \cap J^n\} \) is cofinal with \( \{(I + J)^n\} \), and since the system \( \{I^n + J^n\} \) is cofinal with \( \{(I + J)^n\} \), we can take direct limits over these long exact sequences to get the desired long exact sequence. ∎

3. Computing local cohomology

One disadvantage of working with derived functors is that computing injective resolutions in practice is rather difficult. We already saw how the long exact sequence associated to a short exact sequence can be very useful, since we can break apart local cohomology modules into pieces we can understand. We first give another example where a similar breaking apart procedure is useful.

3.1. The Mayer–Vietoris sequence. In algebraic topology, we learn about the Mayer–Vietoris sequence, which allows us to break apart a topological space into smaller pieces whose (co)homology we hopefully understand. We have a similar result for local cohomology:

\[ C(x_1, \ldots, x_t; M) := \left\{ 0 \rightarrow M \xrightarrow{d} \bigoplus_{i=1}^{t} M[x_i^{-1}] \xrightarrow{d} \bigoplus_{|J|=s} M[x_J^{-1}] \xrightarrow{d} \cdots \bigoplus M[x_{\{1, \ldots, t\}}^{-1}] \rightarrow 0 \right\} \]
where the differential takes an element
\[ m_J \in M[x_J^{-1}] \subseteq \bigoplus_{|J|=s} M[x_J^{-1}] \]
to the element
\[ d(m_J) = \sum_{k \notin J} (-1)^{o_J(k)} m_{J \cup \{k\}}, \]
where \( o_J(k) \) denotes the number of elements of \( J \) less than \( k \), and \( m_{J \cup \{k\}} \) denotes the image of \( m_J \) in the further localization \( M[x_J^{-1}] = M[x_J^{-1}][x_k^{-1}] \).

We omit the proof, and instead point the reader to [Eis05, Thm. A1.3] or [Hun07, Prop. 2.13].

**Corollary 9.** If \( I = (x_1, \ldots, x_t) \), then \( H^1_I(M) = 0 \) for all \( i > t \).

*Proof.* The length of the Čech complex \( C(x_1, \ldots, x_t; M) \) is \( t \).

We have therefore found a nice functor that satisfies the properties we wanted!

4. **Hartshorne’s Example**

We now turn back to Example 2.

**Exercise 10** (Hartshorne’s Example [Har70 Exc. III.5.12; Har77 Exc. III.4.9]). Consider the ideal
\[ I(Y) = (x, y) \cap (u, v) = (xu, xv, yu, yv) \subseteq k[x, y, u, v]. \]
We want to show that \( I(Y) \) cannot be generated, up to radical, by two elements.

1. Using our two main properties, show that it suffices to show \( H^3_I(Y)(R) \neq 0 \).

   *Proof.* Suppose \( J \) is an ideal generated by two elements such that \( \sqrt{J} = I(Y) \). Then, by Corollary 9 of the identification of local cohomology with Čech cohomology, we have \( H^3_J = H^3_I(Y) = 0 \), using our other main property that local cohomology does not change under taking the radical of \( J \). \( \square \)

2. Use the Mayer–Vietoris sequence on \( I(Y) \) and the vanishing property to show \( H^3_I(Y)(R) \cong H^4_m(R) \), where \( m = (x, y) + (u, v) \).

   *Proof.* The Mayer–Vietoris sequence gives
   \[ H^3_{(x,y)}(R) \oplus H^3_{(u,v)}(R) \rightarrow H^3_I(Y)(R) \rightarrow H^4_m(R) \rightarrow H^4_{(x,y)}(R) \oplus H^4_{(u,v)}(R). \]
   The two edge modules vanish since \( (x, y) \) and \( (u, v) \) are generated by two elements each, and by applying Corollary 9 \( \square \)

3. Compute \( H^4_m(R) \) using Čech cohomology. *Hint:* The relevant part of the Čech complex is

\[
\begin{array}{c}
\oplus \\
\oplus \\
\oplus \\
\oplus \\
\end{array}
\xrightarrow{1}
\xrightarrow{-1}
\xrightarrow{1}
\xrightarrow{-1}
\rightarrow 0
\]

*Proof.* The hard part is writing down the complex. Using the above representation of the complex, we obtain
\[ H^3_I(Y)(R) \cong H^4_m(R) \cong k(a^b \cdot x^c \cdot x^d \mid a, b, c, d < 0) \neq 0. \] \( \square \)
5. Back to Curves

In Example 1, we saw that it was fairly easy to construct examples of non-complete intersections by considering monomial curves in $\mathbb{P}^3$. The twisted cubic ended up being a set-theoretic complete intersection. However, the generalization of this fact is not known:

**Open Question** ([Har77, Exc. I.2.17(d)]). *Can all irreducible curves in $\mathbb{P}^3$ be defined set-theoretically by two equations?*

This question is surprisingly difficult: while local cohomology gives us an obstruction for a variety to be a set-theoretic complete intersection, if the relevant local cohomology module vanishes, we get no information. Even for the following simple example, we have no idea:

**Example 11.** Consider the smooth rational quartic curve $C$ in $\mathbb{P}^3$ defined as the image of the map

$$\mathbb{P}^1 \to \mathbb{P}^3$$

$$[s : t] \mapsto [s^4 : s^3t : st^3 : t^4]$$

It is known that $H^i_{\mathcal{I}(C)}(M) = 0$ for all $i > 2$ and all modules $M$ [Har70, Ch. III], so local cohomology does not provide an obstruction to $C$ being a set-theoretic complete intersection. On the other hand, it is known that if we work over a field of characteristic $p > 0$, we do get a set-theoretic complete intersection [Har79].

References


