FINITE GENERATION OF $K$-GROUPS OF RINGS
OF INTEGERS IN NUMBER FIELDS

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Abstract. I present Quillen’s proof of the fact that the $K$-groups $K_i A$ where $A$ is the ring of integers in a number field $F$ are finitely generated. This talk was given in the Algebraic $K$-theory seminar at Michigan during the Winter semester of 2015.

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INTRODUCTION

We follow [Qui73a]. The main result we will show in this talk is the following

Theorem 1. If $A$ is the ring of algebraic integers in a number field $F$ (finite over $Q$) then $K_i A$ is a finitely generated group for all $i \geq 0$.

To show this, we use the definition of $K_i A$ as

$$K_i A := \pi_{i+1}(N(Q \mathcal{P}), 0)$$

where $Q-$ denotes Quillen’s $q$-construction [Qui73b], $N-$ denotes the nerve of a category, and $\mathcal{P}$ is the category of finitely-generated projective $A$-modules.

We pause to note the arithmetic significance of this result. Lichtenbaum in [Lic73] conjectured that higher $K$-groups $K_i A$ can give information about special values of the Dedekind zeta functions $\zeta_F(s)$. In particular, the ranks of $K_i A$ computed by Borel in [Bor74] give multiplicities of trivial zeros of $\zeta_F(s)$. We unfortunately don’t have much time to go deeply into this, but see [Kah05, Wei05] for surveys on the subject.

1. Buildings

To prove Theorem 1, we introduce the notion of a building.

Definition. The building $[V]$ of an $n$-dimensional vector space $V$ over a field $F$ is the nerve associated to the poset of nontrivial proper subspaces of $V$, i.e., $p$-simplices are chains $0 \subsetneq W_0 \subsetneq \cdots \subsetneq W_p \subsetneq V$ of subspaces $W_i$ of $V$. If $n \leq 1$, then $[V] = \emptyset$; if $n = 2$, then $[V]$ is the projective space $P(V)$ of lines in $V$ as a discrete space.

A fundamental theorem about buildings is the following

Theorem 2 (Solomon-Tits [Sol69]). If $n \geq 2$, then $[V]$ has the homotopy type of a wedge of $(n-2)$-spheres.
To prove this, we first state a corollary of the following theorem by Quillen:

**Theorem A*.** Let $f : \mathscr{C} \to \mathscr{C}'$ be a functor, and let $f/Y := \{(X, u) \mid u: f(X) \to Y\}$. Then, if $N(f/Y)$ is contractible for every object $Y \in \mathscr{C}'$, then $f$ induces a homotopy equivalence of nerves.

**Lemma 3.** Let $g : K \to K'$ be a simplicial map of simplicial complexes. If for every closed subsimplex $\sigma \subset K$, its inverse image $g^{-1}(\sigma)$ is contractible, then $g$ is a homotopy equivalence.

**Proof.** Let $\text{Simpl} K$ and $\text{Simpl} K'$ denote the posets of subsimplices of $K,K'$ ordered by inclusion, and let $h$ be the map on posets such that $g$ is homeomorphic to $Nh$. Then, $h/\sigma$ is the poset of subsimplices of $g^{-1}(\sigma)$, and the claim follows by Theorem A*. □

Now to prove the Solomon-Tits theorem, we introduce a new simplicial complex. We call $V$ the simplicial complex with $p$-simplices being chains $0 \subseteq W_0 \subsetneq \cdots \subsetneq W_p \subseteq V$, i.e., it is defined in the same way as $\text{V}$ but $W_0$ can be 0. Note $\text{Cone} \text{V} \simeq \text{V}$.

**Proof of Theorem 2 (by induction).** If $n = 2$, then the claim is trivial since $\text{V}$ is discrete hence is trivially a wedge of 0-spheres.

Now suppose $n \geq 3$. Fix a line $L \subseteq V$, and let $H$ be the set of hyperplanes $H$ such that $V = H \oplus L$. Now let $Y$ be the full subcomplex of $\text{V}$ obtained by removing the vertices $H$.

**Claim.** $Y$ is contractible.

**Proof of Claim.** If $V \to V/L$ is the projection, we get an induced simplicial map $q : Y \to \overline{V/L}$; the latter is contractible since it has a minimal element 0. Now by the Lemma 3, it suffices to show for every subsimplex $\sigma = (W_0/L \subsetneq \cdots \subsetneq W_p/L)$ of $\overline{V/L}$, its inverse image $q^{-1}(\sigma)$ in $Y$ is contractible.

Now if $U \in q^{-1}(\sigma)$, then $q(U) = W_i/L$ for some $i$, so $U + L = W_i$. We can then visualize the simplices in $Y$ and $\overline{V/L}$ as follows:

```
Y  \quad U_0 + L \subsetneq U_1 + L \subsetneq \cdots \subsetneq U_p + L
q  \downarrow
  \quad U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_p
\overline{V/L} \quad \sigma = W_0/L \subsetneq W_1/L \subsetneq \cdots \subsetneq W_p/L
```

“Pushing up” then defines a deformation retraction $q^{-1}(\sigma) \simeq \Delta^p$. □

We then have the following schematic picture of $\text{V}$ from [Qui10, p. 483]:

![Diagram of the simplicial complex V](image-url)
where for each \( H \in \mathcal{K} \), \( \text{Link}(H) \) is the subcomplex of \([V]\) formed by simplices \( \sigma \) such that \( H \notin \sigma \) but \( \sigma \cup \{H\} \) is a simplex. Note \( \text{Link}(H) \subset Y \), and that \([V]\) is the union of \( Y \) with the cones over these links, glued along the \( \text{Link}(H) \) as \( H \) varies in \( \mathcal{K} \). Thus,

\[
[V] \simeq [V]/Y \simeq \bigvee_{H \in \mathcal{K}} \text{Link}(H).
\]

Now \( \text{Link}(H) = \overline{H} \) for any \( H \in \mathcal{K} \), so the theorem follows by induction since \( \dim H = n - 1 \).

Now we define another poset \( J(V) \) which will be useful later because it is simpler to analyze.

Let \( J(V) \) be the set of subspaces \( W_0 \subseteq W_1 \) of \( V \) such that \( \dim(W_1/W_0) < n \), ordered by \( (W_0, W_1) \preceq (W'_0, W'_1) \) if \( W'_0 \preceq W_0 \) and \( W_1 \preceq W'_1 \). For \( n = 1 \), \( J(V) \) consists of \((0, 0)\) and \((V, V)\), which are incomparable, hence \( NJ(V) = S^0 \).

**Proposition.** If \( n \geq 2 \), there is a \( \text{GL}(V) \)-equivariant homotopy equivalence

\[
\overline{V} \to N(J(V))
\]

where \( \overline{V} \) is the simplicial complex with \( p \)-simplices being chains \( 0 \subseteq W_0 \subseteq \cdots \subseteq W_p \subseteq V \) such that \( \dim(W_p/W_0) < n \), which is a subsimplicial complex of the complex \( \overline{V} \) formed without this restriction.

**Proof.** Define a map \( g: \text{Simpl} \overline{V} \to J(V) \) by

\[
g(W_0 \subseteq W_1 \subseteq \cdots \subseteq W_p) = (W_0, W_p).
\]

\( g \) is a \( \text{GL}(V) \)-equivariant functor. Now \( N(\text{Simpl} \overline{V}) \) is a barycentric subdivision of \( \overline{V} \), so it suffices to show \( N \circ g \) is a homotopy equivalence. By Theorem A*, it suffices to show for each \( (U_0, U_1) \in J(V) \), the category \( g/(U_0, U_1) \) is contractible. But the objects of \( g/(U_0, U_1) \) are simplices \( W_0 \subseteq W_1 \subseteq \cdots \subseteq W_p \) such that \( U_0 \nsubseteq W_0 \) and \( W_p \nsubseteq U_1 \) ordered by inclusion. Now this is isomorphic to \( \text{Simpl} \overline{V} \), which is contractible since it has an initial object.

Noting that \( \overline{V} \simeq \bigvee \overline{V} \), we have the following

**Corollary 4.** Suppose \( n \geq 1 \). The reduced homology \( \tilde{H}_i(NJ(V)) \) vanishes for \( i \neq n - 1 \), and is a free \( \mathbb{Z} \)-module for \( i = n - 1 \).

**Definition.** The \( \mathbb{Z} \)-module \( \tilde{H}_{n-1}(NJ(V)) \cong \tilde{H}_{n-2}(\overline{V}) \) with the natural \( \text{GL}(V) \) action is called the **Steinberg module** of \( V \), and is denoted \( \text{st}(V) \).

## 2. A Long Exact Sequence

The main homological input in this theorem is a long exact sequence, which we will prove in this section.

Let \( A \) be a Dedekind ring with field of fractions \( F \). For each \( n \geq 0 \), let \( Q_n \) be the full subcategory of \( Q \mathcal{P}(A) \) formed by projective modules of rank \( \leq n \). Then, \( Q_0 \) is the trivial category with one object and no morphisms, \( Q_n \subset Q_{n+1} \), and \( Q = \bigcup_n Q_n \).

**Theorem 5.** Let \( n \geq 1 \). The inclusion \( w: Q_{n-1} \to Q_n \) induces a long exact sequence

\[
\cdots \to H_i(NQ_{n-1}) \to H_i(NQ_n) \to \prod_{\alpha} H_{i-n}(\text{GL}(P_\alpha), \text{st}(V_\alpha)) \to H_{i-1}(NQ_{n-1}) \to \cdots
\]

where \( P_\alpha \) represent isomorphism classes of projective modules of rank \( n \), and \( V_\alpha = P_\alpha \otimes_A F \).

To prove this theorem, we first recall the following fact about computing homology of nerves:
Proposition 6 ([GZ67 App. II, 3.3]). If $F: C \to \text{Ab}$ is a functor, then $H_n(NC, F)$ can be computed by $\lim_{\to}^\ell F$, the $n$th derived functor of $\lim_{\to}^\ell: \text{Ab}^C \to \text{Ab}$.

Example. If $L$ is the constant functor $\mathbf{Z}$, then $\lim_{\to}^\ell \mathbf{Z}$ gives integral homology $H_n(NC)$.

Proof of Theorem 5. The main ingredient in this proof is a Grothendieck spectral sequence obtained from the following commutative diagram of functors:

\[
\text{Ab}^{Q_{n-1}} \xrightarrow{w^*} \text{Ab}^{Q_n} \\
\text{Ab} \xrightarrow{\lim_{\to} Q_{n-1}} \text{Ab} \\
\text{Ab}^{(w/P)} \xrightarrow{\lim_{\to} (w/P)} \text{Ab}
\]

where $w^*$ is defined (on objects) by

\[(w^*f)(P) = \lim_{(P', u) \in \text{w/P}} f(P') = \lim_{u(P') \to P} f(P')\]

for any functor $f: Q_{n-1} \to \text{Ab}$. Note this “looks like” the pullback functor for sheaves, and indeed the functor $w_*$ defined by precomposition is a right adjoint to this functor that preserves epimorphisms since it does so componentwise. Thus, $w^*$ preserves projectives and so we have the following Grothendieck spectral sequence:

\[E_2^{pq} = \lim_{\to}^{Q_n}(Lq(w^*)(f)) \Rightarrow \lim_{\to}^{Q_{n-1}}(f)\]

for any functor $f: Q_{n-1} \to \text{Ab}$. We can simplify this further by looking at the following commutative diagram:

\[
\text{Ab}^{Q_{n-1}} \xrightarrow{w^*} \text{Ab}^{Q_n} \\
i_{P*} \downarrow \quad \downarrow \text{eval}_P \\
\text{Ab}^{(w/P)} \xrightarrow{\lim_{\to} (w/P)} \text{Ab}
\]

for each $P \in Q_n$, where $i_P$ is the projection $w/P \to Q_{n-1}$ defined by $(P', u) \mapsto P'$. The two vertical functors are exact, hence we have that

\[(Lq(w^*)(f)) \cong (P \mapsto \lim_{\to} w/P f \circ i_P)\]

and the spectral sequence becomes

\[E_2^{pq} = \lim_{\to}^{Q_n}(P \mapsto \lim_{\to}^{(w/P)} f \circ i_P) \Rightarrow \lim_{\to}^{Q_{n-1}}(f).\]

In particular, for $f = \mathbf{Z}$, by Proposition 6 this becomes the spectral sequence

\[E_2^{pq} = \lim_{\to}^{Q_n}(P \mapsto H_q(N(w/P))) \Rightarrow H_{p+q}(NQ_{n-1}). \quad (1)\]

Now to use this spectral sequence, we would like to know what $H_q(N(w/P))$ looks like. Recall $w/P = \{(P', u) \mid u: P' \to P\}$. We can write down the following bijection, noting that the morphisms $u: P' \to P$ in $Q_n$ are of the form on the left by the $q$-construction:

\[
\{P' \leftarrow P_1 \Rightarrow P\} \leftrightarrow \left\{ \text{pairs } (P_0, P_1) \text{ of submodules } P_0 \subseteq P_1 \text{ of } P \\right\} \\
\text{such that } P' \cong P_1/P_0 \text{ is an isomorphism}
\]

so $w/P$ is equivalent to the poset $J$ of pairs of submodules $P_0 \subseteq P_1$ of $P$ such that $\text{rk}(P_1, P_0) < n$, with the ordering $(P_0, P_1) \leq (P'_0, P'_1)$ if $P'_0 \subseteq P_0$ and $P_1 \subseteq P'_1$. 

Now if \( \operatorname{rk} P < n \), then \( J \) has a maximal element \((0, P)\), so \( N(w/P) \) is contractible. If instead \( \operatorname{rk} P = n \), then the map \( P' \to P' \otimes_A F \subset V = P \otimes_A F \) induces an equivalence \( J \simeq J(V) \), hence Corollary 4 applies. Thus, if \( n = 1 \), then

\[
H_q(N(w/P)) = 0 \quad \text{if } q > 0
\]
\[
H_0(N(w/P)) = \begin{cases} \mathbb{Z} & \text{if } P = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } \operatorname{rk} P = 1 \\ \end{cases}
\]

and if \( n \geq 2 \),

\[
H_0(N(w/P)) = \mathbb{Z}
\]
\[
H_q(N(w/P)) = 0 \quad \text{if } q \neq 0, n - 1
\]
\[
H_{n-1}(N(w/P)) = \begin{cases} 0 & \text{if } \operatorname{rk} P < n \\ \text{st}(V) & \text{if } \operatorname{rk} P = n \\ \end{cases}
\]

We first analyze the case when \( n \geq 2 \). The \( E^2_{pq} \) terms in the spectral sequence (1) are given by

\[
E^2_{pq} = \lim_{\to}^{Q_n}(P \mapsto H_q(N(w/P))) = \begin{cases} H_p(NQ_n) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0, n - 1 \\ \end{cases}
\]

It remains to analyze what happens when \( q = n - 1 \). To do so, consider the full subcategory \( Q' \) of \( Q_n \) consisting of rank \( n \) projectives. Since functor \( P \mapsto H_q(N(w/P)) \) is the zero functor for objects not in \( Q' \), we have an isomorphism of functors

\[
\lim_{\to}^{Q_n}(P \mapsto H_q(N(w/P))) \cong \lim_{\to}^{Q'}(P \mapsto H_q(N(w/P)))
\]

where in the latter, the functor is restricted to \( P \in Q' \) (to be precise, it is necessary to understand the details of the calculation of left-derived functors \( \lim_{\to} \) from [GZ67, App. II, 3.2]). \( Q' \) is equivalent to the groupoid of rank \( n \) projectives and their isomorphisms, which is in turn equivalent to the full skeletal subcategory with one object \( P_\alpha \) from each isomorphism class, and this is the category for the groupoid \( Q'' = \coprod_\alpha \text{GL}(P_\alpha) \). On this category, the functor \( P_\alpha \mapsto H_{n-1}(w/P) \) maps \( P_\alpha \) to the \( \text{GL}(P_\alpha) \)-modules \( \text{st}(V_\alpha) \) where \( V_\alpha = P_\alpha \otimes_A F \) by (2). Thus,

\[
E^2_{p,n-1} = \lim_{\to}^{Q_\alpha}(P \mapsto H_{n-1}(N(w/P))) = \coprod_\alpha H_p(\text{GL}(P_\alpha), \text{st}(V_\alpha)) =: L_p
\]

and the \( E^2 \) page of our spectral sequence looks like

\[
\begin{array}{cccccc}
\ldots & L_0 & L_1 & \cdots & L_p & \cdots \\
\vdots & \cdots & 0 & \cdots & \cdots & \cdots \\
0 & H_0(NQ_n) & H_1(NQ_n) & \cdots & H_p(NQ_n) & \cdots \\
\end{array}
\]
which is the same until the $E^n$ page, where it converges to $E^\infty$. The desired long exact sequence is obtained by splicing together the maps from the $E^\infty$ page:

$$0 \to E_{p+1,0}^\infty \to H_{p+1}(NQ_n) \to L_{p+1-n} \to H_p(NQ_{n-1}) \to H_p(NQ_n) \to L_{p-n} \to \cdots$$

Finally, for $n = 1$, the spectral sequence (1) degenerates to an isomorphism

$$\lim_{\to}^Q (P \mapsto H_0(w/P)) \cong H_p(Q_0) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

We also have the short exact sequence

$$0 \to (P \mapsto \tilde{H}_0(w/P)) \to (P \mapsto H_0(w/P)) \to \mathbb{Z} \to 0$$

in $\text{Ab}^Q$ and just like the case $n \geq 2$, we have the isomorphism

$$\lim_{\to}^Q (P \mapsto H_0(N(w/P))) = \prod_{\alpha} H_p(GL(P_\alpha), \text{st}(V_\alpha)) =: L_p$$

The desired long exact sequence is exactly that obtained by the long exact sequence on homology for $\lim_{\to}^Q$:

$$\cdots \to H_p(NQ_0) \to H_p(NQ_1) \to L_{p-1} \to H_{p-1}(NQ_0) \to \cdots \square$$

3. **Proof of Theorem 1**

3.1. **Reduction to a Group Homology Calculation.** We are now ready to prove the main theorem. Grayson in [Gra82] noted that Quillen’s argument boils down to the following statement:

**Proposition 7.** Let $A$ be a Dedekind domain with fraction field $F$. Then, $K_i A$ is finitely generated for all $i \geq 0$ if Pic $A$ is finite and $H_i(\text{GL}(P), \text{st}(V))$ is finitely generated for all $P \in \mathcal{P}$ and $V = P \otimes_A F$.

**Proof.** By Theorem 5, there is a long exact sequence

$$\cdots \to H_i(NQ_{n-1}) \to H_i(NQ_n) \to \prod_{\alpha} H_{i-n}(\text{GL}(P_\alpha), \text{st}(V_\alpha)) \to H_{i-1}(NQ_{n-1}) \to \cdots$$

Note for each $n$ there are only finitely many isomorphism classes of projective $A$-modules of rank $n$ by the finiteness of Pic $A$ and the classification of finitely generated modules over a Dedekind domain [DF04, Ch. 6, Thm. 22], so the groups $\prod_{\alpha} H_{i-n}(\text{GL}(P_\alpha), \text{st}(V_\alpha))$ are finitely generated.

We first claim $H_i(NQ_n)$ is finitely generated for all $i, n$. First, $H_0(NQ_0) = \mathbb{Z}$ and $H_i(NQ_0) = 0$ for all $i > 0$ by the fact that $Q_0$ is the trivial category. Using the long exact sequence above, the claim follows by induction.
We outline the construction of such a classifying space in the sequel.

We give a brief outline of the construction.

Theorem 8. If $\sigma_i \in \mathcal{P}(A)$ and $V = P \otimes_A F$, then $H_i(\text{GL}(P), \text{st}(V))$ is finitely generated for all $i$. 

Note that $\text{GL}(P)$ is an arithmetic subgroup of $\text{GL}(V \otimes \mathbb{Q} R)$, so the main technical tool we use here is the Borel-Serre compactification, which is used to compute the (co)homology of arithmetic groups.

3.2. The Borel-Serre compactification. We give a short description of the compactification and its applications to computing (co)homology of arithmetic groups in our particular case; see [Bor73, Bor06] for surveys on the cohomology of arithmetic groups, and [Gor05, Sap03] for surveys on the Borel-Serre (and other compactifications) in the reductive case. Note that most treatments of the Borel-Serre compactification only deal with the semi-simple case.

Let $G$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$; we write $G = G(\mathbb{R})$. Fix an arithmetic subgroup $\Gamma \subset G(F)$ where $F$ is a number field, that is, a subgroup a subgroup that is commensurable with $G(A)$ where $A$ is the ring of integers in $F$. We make the assumption that $\Gamma$ is torsion-free. One motivation for the Borel-Serre compactification is the following

Question. What is a good finite classifying space $B\Gamma$ for an arithmetic group $\Gamma$, from which we can deduce facts about the group cohomology of $\Gamma$?

We outline the construction of such a classifying space in the sequel.

Let $K \subset G$ be a fixed maximal compact subgroup, and let $A_G$ be the (topologically) connected identity component of the group of real points of the greatest $\mathbb{Q}$-split torus $A_G$ in the center of $G$. Define $X := G/KA_G$. Then, Borel and Serre proved the following

Theorem 9 ([BS73]). There exists an enlargement $\overline{X}$ of $X$ satisfying the following properties:

1. $\overline{X}$ is contractible [BS73, Lem. 8.6.4];
2. $\partial \overline{X}$ has the homotopy type of the Tits building $T$ associated to $G(\mathbb{Q})$, and has dimension $\ell$, the $\mathbb{Q}$-rank of $G/RG$, where $RG$ denotes the radical of $G$ [BS73, Thm. 8.4.1];
3. $\Gamma$ acts freely on $\overline{X}$, hence $\overline{X}/\Gamma$ is a space $B\Gamma$ that is compact [BS73, Thm. 9.3, n° 9.5].

We give a brief outline of the construction.

Construction. First, for any $G$ as above, define $^0G$ as follows:

$$^0G := \bigcap \ker(\chi^2),$$

where the intersection is taken over all rationally defined characters $\chi: G \to G_m$ [BS73, n° 1.1].

Then, the group $G$ decomposes as $G = ^0GA_G$, where $^0G = ^0G(\mathbb{R})$ [BS73, Prop. 9.2].
Let $P$ be a rational parabolic subgroup of $G$, and let $P = P(R)$ be its group of real points. Then, $P$ has a Langlands decomposition

$$P = U_P A_P L_P,$$

where $U_P$ is the unipotent radical of $P$, $L_P$ is the Levi quotient of $P$, and the decomposition follows as above. Now define $e(P) = X/A_P$. $X$ is obtained set-theoretically by glueing each space $e(P)$ to $X$, and the topology is defined in [BE73, §7]. The closures of the $e(P)$ form a locally finite cover of the boundary $\partial X$, whose nerve is the Tits building [BS73, Thm. 8.4.1]. The nerve of a locally finite cover of a space is homotopy equivalent to the space itself by [BS73, Thm. 8.2.2], so we have $(b)$. □

3.3. Proof of Theorem 8. We are now ready to prove Theorem 8. Recall that $A$ is the ring of integers in a number field $F$, and $P$ is a finitely-generated projective module over $A$.

Let $G$ be the general linear group, such that $G(F) = \text{GL}(V)$, where $V = P \otimes_A F$. By [BS73, Thm. 8.2.2], $\partial X$ has the homotopy type of the Tits building $T$ associated to $\text{GL}(V)$. In our case, there is a natural isomorphism $V \to T$ where the simplex $W_0 \subseteq \cdots \subseteq W_p$ of $V$ corresponds to the reverse chain of stabilizers of each space in $\text{GL}(V)$. Its homology is then described by Theorem 2.

Now we can compute the cohomology (with compact support) of $X$.

**Proposition 10 ([BS73, Thm. 8.6.5]).** The groups $H^i_c(X)$ are 0 for $i \neq d - \ell$, where $d = \dim X$. The group $H_c^{d - \ell}(X)$ is free abelian, and is isomorphic to $\text{st}(V)$.

**Proof.** If $\ell = 0$, $X = X$, hence is a orientable manifold, and Poincaré duality gives that

$$H^i_c(X) \cong H_{d-i}(X)$$

and the result follows by contractibility of $X$.

For $\ell \geq 1$, since $X$ is contractible, the long exact sequence of the pair gives isomorphisms $\tilde{H}_j(\partial X) \cong H_{j+1}(X, \partial X)$. Then, Poincaré duality for manifolds with boundary [Hat02, Thm. 3.35] gives isomorphisms

$$H^j_c(X) \cong H_{d-i}(X, \partial X) \cong H_{d-i-1}(\partial X)$$

and the result follows by Theorem 2 and Theorem 9. □

Now consider the arithmetic subgroup $G(L)$ of $G(V) = G(F)$. It has a finite index torsion-free subgroup $\Gamma$ by Minkowski’s theorem (see [Sou07, Thm. 8] for a proof). Then, $X/\Gamma$ is a classifying space $BT$, hence we can calculate the cohomology of $\Gamma$ by calculating the cohomology of this space. We then have the following

**Theorem 11 (Duality, [BS73, Thm. 11.4.2]).** There is an isomorphism

$$H^{d - \ell - i}(\Gamma, Z) \cong H_i(\Gamma, \text{st}(V))$$

for each $i$.

**Proof.** General facts about classifying spaces in, say, [BE73, n° 6.3] imply the isomorphisms

$$H^i(\Gamma, Z[\Gamma]) \cong H^i_c(X/\Gamma, Z[\Gamma]) \cong H^i_c(X, Z)$$

hence $H^i(\Gamma, Z[\Gamma]) = 0$ for all $i \neq d - \ell$, and $H^{d - \ell}(\Gamma, Z[\Gamma]) \cong \text{st}(V)$ is free abelian by Proposition 10. $\Gamma$ is thus a duality group in the sense of [BE73, Thm. 4.5], and the isomorphism follows. □

By [BS73, n° 11.1], $H^{d - \ell - i}(\Gamma, Z)$ is finitely generated for each $i$, hence by the Duality theorem above, $H_i(\Gamma, \text{st}(V))$ is as well. Finally, the homology spectral sequence

$$H_p(\text{GL}(P)/T, H_q(\Gamma, \text{st}(V))) \Rightarrow H_{p+q}(\text{GL}(V), \text{st}(V))$$

implies that since $\text{GL}(V)/\Gamma$ is finite, the groups $H_i(\Gamma, \text{st}(V))$ are finitely generated. □

We finally note that while it is beyond the scope of this talk, we know the ranks of the $K$-groups $K_iA$:
Theorem 12 ([Bor74]). The ranks are given by the formula

\[ \text{rk} K_i A = \begin{cases} 
  r_1 + r_2 & \text{if } n \equiv 1 \pmod{4} \\
  r_2 & \text{if } n \equiv 3 \pmod{4} \\
  0 & \text{otherwise}
\end{cases} \]

where \( r_1, r_2 \) are the numbers of real and complex places in \( A \), respectively.

On the other hand, the torsion parts of \( K_i A \) were not known until recently; see [Wei05].

References


