F-SINGULARITIES FOR NON-F-FINITE RINGS

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We review some classes of singularities defined using the Frobenius morphism, taking care to avoid F-finiteness assumptions. Most of this material is well-known, but some of the implications in Theorem 8 are new, at least in the non-F-finite case. We recommend [TW18] for a survey of F-singularities (mostly in the F-finite setting), and [DS16, §6] and [Has10, §3] as references for the material on strong F-regularity in the non-F-finite setting.

To define different versions of F-rationality, we will need the following:

**Definition 1** [HH90, Def. 2.1]. Let $R$ be a noetherian ring. A sequence of elements $x_1, x_2, \ldots, x_n \in R$ is a *sequence of parameters* if for every prime ideal $p$ containing $(x_1, x_2, \ldots, x_n)$, the images of $x_1, x_2, \ldots, x_n$ in $R_p$ are part of a system of parameters in $R_p$.

We now begin defining different classes of singularities. We start with $F$-singularities defined using tight closure. Recall that if $R$ is a ring, then $R^\circ$ is the complement of the union of the minimal primes of $R$.

**Definition 2** [HH90, Def. 8.2]. Let $R$ be a ring of characteristic $p > 0$, and let $\iota: N \hookrightarrow M$ be an inclusion of $R$-modules. The tight closure of $N$ in $M$ is the $R$-module $N^*_{M} := \{ x \in M : \text{there exists } c \in R^\circ \text{ such that for all } e \gg 0, \ c \otimes x \in \text{im}(\text{id} \otimes \iota): F^eR \otimes_R N \to F^eR \otimes_R M \}$.

We say that $N$ is tightly closed in $M$ if $N^*_{M} = N$.

**Definition 3** ($F$-singularities via tight closure). Let $R$ be a noetherian ring of characteristic $p > 0$. We say that

(a) $R$ is strongly $F$-regular if $N$ is tightly closed in $M$ for every inclusion $N \hookrightarrow M$ of $R$-modules [Hoc07, Def. on p. 166];
(b) $R$ is weakly $F$-regular if $I$ is tightly closed in $R$ for every ideal $I \subseteq R$ [HH90, Def. 4.5];
(c) $R$ is $F$-regular if $R_p$ is weakly $F$-regular for every prime ideal $p \subseteq R$ [HH90, Def. 4.5]; and
(d) $R$ is $F$-rational if $I$ is tightly closed in $R$ for every ideal $I$ generated by a sequence of parameters in $R$ [FW89, Def. 1.10].

We note that (a) is not the usual definition of strong $F$-regularity, although it coincides with the usual definition (Definition 6(a)) for $F$-finite rings; see Theorem 8. We also note that the original definition of $F$-regularity asserted that localizations at every multiplicative set are weakly $F$-regular, but the definition using prime ideals is equivalent by [HH90, Cor. 4.15].

Next, we define $F$-singularities via purity of homomorphisms involving the Frobenius. We recall that a ring homomorphism $\varphi: R \to S$ is pure if the homomorphism

$\varphi \otimes \text{id}: R \otimes_R M \to S \otimes_R M$

is injective for every $R$-module $M$. To simplify notation, we fix the following:

**Notation 4.** Let $R$ be a noetherian ring of characteristic $p > 0$. For every $c \in R$ and every integer $e > 0$, we denote by $\lambda^e_c$ the composition

$R \xrightarrow{F_R} F^e_R \xrightarrow{F_R^e(-c)} F^e_R$
**Definition 5** ($F$-singularities via purity). Let $R$ be a noetherian ring of characteristic $p > 0$. For $c \in R$, we say that $R$ is $F$-pure along $c$ if $\lambda^e_c$ is pure for some $e > 0$. Moreover, we say that

(a) $R$ is $F$-pure regular if it is $F$-pure along every $c \in R^\circ$ [HH94, Rem. 5.3];
(b) $R$ is $F$-pure if it is $F$-pure along $1 \in R$ [HR76, p. 121]; and
(c) $R$ is strongly $F$-rational if for every $c \in R^\circ$, there exists $e_0 > 0$ such that for all $e \geq e_0$, the homomorphism $\lambda^e_c \otimes R/I$ is injective for every ideal $I \subseteq R$ generated by a sequence of parameters in $R$ [V´ el95, Def. 1.2].

The terminology $F$-pure regular is from [DS16, Def. 6.1.1] to distinguish it from the definition using tight closure (Definition 3(a)). $F$-pure regular rings are also called very strongly $F$-regular [Has10, Def. 3.4].

We note that $F$-purity is a local condition [DS16, Lem. 6.1.4(e)]. Strong $F$-regularity is a local condition [Has10, Lem. 3.6], and while it is equivalent to $F$-pure regularity in the local case [Has10, Lem. 3.6], $F$-pure regularity is not known to be a local condition [DS16, Rem. 6.3.3].

Next, we define $F$-singularities via splitting of homomorphisms involving the Frobenius. We use the same notation as for $F$-singularities defined using purity (Notation 4).

**Definition 6** ($F$-singularities via splitting). Let $R$ be a noetherian ring of characteristic $p > 0$. For $c \in R$, we say that $R$ is $F$-split along $c$ if $\lambda^e_c$ splits as an $R$-module homomorphism for some $e > 0$. Moreover, we say that

(a) $R$ is split $F$-regular if it is $F$-split along every $c \in R^\circ$ [HH94, Def. 5.1]; and
(b) $R$ is $F$-split if it is $F$-split along $1 \in R$ [MR85, Def. 2].

The terminology split $F$-regular is from [DS16, Def. 6.6.1]. split $F$-regularity is usually known as strong $F$-regularity in the literature.

Finally, we define $F$-injective singularities.

**Definition 7** [Fed83, Def. on p. 473]. A noetherian local ring $(R, \mathfrak{m})$ of characteristic $p > 0$ is $F$-injective if the $R$-module homomorphism

$$H^i_{\mathfrak{m}}(F): H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(F_*R)$$

induced by Frobenius is injective for all $i$. An arbitrary noetherian ring $R$ of characteristic $p > 0$ is $F$-injective if $R_{\mathfrak{m}}$ is $F$-injective for every maximal ideal $\mathfrak{m} \subseteq R$.

The relationship between these classes of singularities can be summarized as follows:

**Theorem 8.** Let $R$ be a noetherian ring of characteristic $p > 0$. We have the following diagram of implications of properties of $R$:
where “C–M” (resp. “Gor.”) is an abbreviation for Cohen–Macaulay (resp. Gorenstein).

Proof. We first list the implications that are easy or appear in the literature.

<table>
<thead>
<tr>
<th>Implication</th>
<th>Proof</th>
</tr>
</thead>
<tbody>
<tr>
<td>split $F$-regular $\Rightarrow F$-split</td>
<td>Definition</td>
</tr>
<tr>
<td>$F$-regular $\Rightarrow$ weakly $F$-regular</td>
<td>Definition</td>
</tr>
<tr>
<td>weakly $F$-regular $\Rightarrow F$-rational</td>
<td>Definition</td>
</tr>
<tr>
<td>split $F$-regular $\Rightarrow F$-pure regular</td>
<td>split maps are pure</td>
</tr>
<tr>
<td>$F$-split $\Rightarrow$ $F$-pure</td>
<td>split maps are pure</td>
</tr>
<tr>
<td>regular $\Rightarrow$ strongly $F$-regular</td>
<td>[DS16, Thm. 6.2.1]</td>
</tr>
<tr>
<td>$F$-pure regular $\Rightarrow$ strongly $F$-rational</td>
<td>[Has10, Lem. 3.8]</td>
</tr>
<tr>
<td>$F$-pure regular $\Rightarrow$ strongly $F$-rational</td>
<td>[DS16, Rem. 6.1.5]</td>
</tr>
<tr>
<td>strongly $F$-regular $\Rightarrow$ $F$-regular</td>
<td>[Has10, Cor. 3.7]</td>
</tr>
<tr>
<td>weakly $F$-regular $\Rightarrow$ $F$-pure</td>
<td>[FW89, Rem. 1.6]</td>
</tr>
<tr>
<td>$F$-pure $\Rightarrow$ $F$-injective</td>
<td>[Fed83, Lem. 3.3]</td>
</tr>
<tr>
<td>strongly $F$-rational $\Rightarrow$ $F$-rational</td>
<td>[Vé95, Prop. 1.4]</td>
</tr>
<tr>
<td>$F$-rational $\Rightarrow$ normal</td>
<td>[HH94, Thm. 4.2(b)]</td>
</tr>
<tr>
<td>$F$-rational $\Rightarrow$ weakly normal</td>
<td>[Vé95, Prop. 0.10]</td>
</tr>
<tr>
<td>$F$-rational + locally excellent $\Rightarrow$ Cohen–Macaulay</td>
<td>[HH94, Thm. 4.2(c)]</td>
</tr>
<tr>
<td>$F$-rational + image of C–M ring $\Rightarrow$ Cohen–Macaulay</td>
<td>[HH94, Thm. 4.2(c)]</td>
</tr>
<tr>
<td>$F$-rational + $F$-finite $\Rightarrow$ $F$-split</td>
<td>[Has10, Lem. 3.3]</td>
</tr>
<tr>
<td>$F$-pure + $F$-finite $\Rightarrow$ $F$-split</td>
<td>[HR76, Cor. 5.3]</td>
</tr>
<tr>
<td>$F$-pure + complete local $\Rightarrow$ $F$-split</td>
<td>[Fed83, Lem. 1.2]</td>
</tr>
<tr>
<td>$F$-rational + Gorenstein $\Rightarrow$ $F$-regular</td>
<td>[HH94, Cor. 4.7(a)]</td>
</tr>
<tr>
<td>$F$-injective + quasi-Gorenstein $\Rightarrow$ $F$-pure</td>
<td>[EH08, Rem. 3.8]</td>
</tr>
</tbody>
</table>

We now show the remaining implications, for which we could not find a reference.

$F$-pure $\Rightarrow$ weakly normal. We adapt the proof of [Sch09, Thm. 4.7]. It suffices to show that if $R$ is $F$-pure, then $R_p$ is weakly normal for every prime ideal $p \subseteq R$ by [Man80, IV.4]. Suppose not, and choose a prime ideal $p \subseteq R$ of minimal height such that $R_p$ is not weakly normal. The local ring $R_p$ is $F$-pure by [DS16, Lem. 6.1.4(e)] hence $F$-injective and reduced. Moreover, the punctured spectrum $\text{Spec}(R_p) \setminus \{pR_p\}$ is weakly normal by the minimality of $p$, hence [Sch09, Lem. 4.6] implies $R_p$ is weakly normal, a contradiction.

Weakly $F$-regular + Gorenstein away from isolated points + Cohen–Macaulay $\Rightarrow$ strongly $F$-regular. Let $R$ be the weakly $F$-regular ring that is Cohen–Macaulay, and also Gorenstein away from isolated points. Then, the localization $R_m$ is weakly $F$-regular for every maximal ideal $m \subseteq R$ by [HH90, Cor. 4.15], and to show that $R$ is strongly $F$-regular, it suffices to show that 0 is tightly closed in

$$E_m := E_{R_m}(R/m)$$
for every maximal ideal $m \subseteq R$ [Has10, Lem. 3.6]. Since $R_m$ is weakly $R$-regular, every submodule of a finitely generated module is tightly closed [HH90, Prop. 8.7], hence the finitistic tight closure $0^{fg}_{E_m}$ as defined in [HH90, Def. 8.19] is zero. Finally, since $0^{fg}_{E_m} = 0_{E_m}$ under the assumptions on $R$ [LS01, Thm. 8.8], we see that $0$ is tightly closed in $E_m$, hence $R$ is strongly $F$-regular.

**Remark.** When the ring is $\text{I. M. Aberbach. Some conditions for the equivalence of weak and strong } F$-regularity. In the implication $F$-regular $\iff$ strongly $F$-regular. We adapt the proof of [LS99, Cor. 4.4]. Let $R$ be the $\mathbb{N}$-graded ring with irreducible ideal $m$; note that by assumption in [LS99, §3], the ring $R$ is finitely generated over a field $R_0 = k$ of characteristic $p > 0$. The localization $R_m$ of $R$ is weakly $F$-regular by [HH90, Cor. 4.15]. Now let $L$ be the perfect closure of $k$, and let $m'$ be the expansion of $m$ in $R \otimes_k L$; since $R$ is graded, $m'$ is the irrelevant ideal in $R \otimes_k L$. The ring homomorphism

$$R_m \to R_m \otimes_k L \cong (R \otimes_k L)_{m'}$$

is purely inseparable and $m$ expands to $m'$, hence $(R \otimes_k L)_{m'}$ is weakly $F$-regular by [HH94, Thm. 6.17(b)]. By the proof of [LS99, Cor. 4.3], $R \otimes_k L$ is strongly $F$-regular. Finally, $R$ is a direct summand of $R \otimes_k L$ as an $R$-module, hence $R$ is strongly $F$-regular as well [Has10, Lem. 3.17].

**F-rational $+$ $F$-finite $\Rightarrow$ strongly $F$-rational.** The hypotheses of [V´el95, Thm. 1.12] are satisfied when the ring is $F$-finite since an $F$-finite ring is excellent and is isomorphic to a quotient of a regular ring of finite Krull dimension by [Gab04, Rem. 13.6].

**Remark.** The condition that $R$ is the image of a Cohen–Macaulay ring is not too restrictive in practice. For instance, it suffices for $R$ to have a dualizing complex [Kaw02, Cor. 1.4], which in turn is implied by $F$-finiteness [Gab04, Rem. 13.6].

**Remark.** In the implication $\text{Weakly } F$-regular $+$ Gorenstein away from isolated points $+$ Cohen–Macaulay $\Rightarrow$ strongly $F$-regular, MacCrimmon [Mac96, Thm. 3.3.2] showed that for $F$-finite rings, the Gorenstein condition can be weakened to being $\mathbb{Q}$-Gorenstein away from isolated points. The implication weakly $F$-regular $+$ $F$-finite $\Rightarrow$ split $F$-regular is a famous open problem, and is known in dimensions at most three by [Wil95, §4]. See also [Abe02] for other situations in which this implication is known and for a proof of MacCrimmon’s theorem.

**References**


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