We review some classes of singularities defined using the Frobenius morphism, taking care to avoid $F$-finiteness assumptions. Most of this material is well-known, but some of the implications in Theorem 8 are new, at least in the non-$F$-finite case. We recommend [TW18] for a survey of $F$-singularities (mostly in the $F$-finite setting), and [DS16, §6] and [Has10, §3] as references for the material on strong $F$-regularity in the non-$F$-finite setting.

To define different versions of $F$-rationality, we will need the following:

**Definition 1** [HH90, Def. 2.1]. Let $R$ be a noetherian ring. A sequence of elements $x_1, x_2, \ldots, x_n \in R$ is a sequence of parameters if for every prime ideal $p$ containing $(x_1, x_2, \ldots, x_n)$, the images of $x_1, x_2, \ldots, x_n$ in $R_p$ are part of a system of parameters in $R_p$.

We now begin defining different classes of singularities. We start with $F$-singularities defined using tight closure. Recall that if $R$ is a ring, then $R^0$ is the complement of the union of the minimal primes of $R$.

**Definition 2** [HH90, Def. 8.2]. Let $R$ be a ring of characteristic $p > 0$, and let $\iota: N \hookrightarrow M$ be an inclusion of $R$-modules. The tight closure of $N$ in $M$ is the $R$-module

$$N^*_M := \left\{ x \in M \mid \text{there exists } c \in R^c \text{ such that for all } e \gg 0,\ c \otimes x \in \text{im}(\text{id} \otimes \iota: F^e R \otimes_R N \to F^e R \otimes_R M) \right\}.$$  

We say that $N$ is tightly closed in $M$ if $N^*_M = N$.

**Definition 3** ($F$-singularities via tight closure). Let $R$ be a noetherian ring of characteristic $p > 0$. We say that

(a) $R$ is strongly $F$-regular if $N$ is tightly closed in $M$ for every inclusion $N \hookrightarrow M$ of $R$-modules [Hoc07, Def. on p. 166];
(b) $R$ is weakly $F$-regular if $I$ is tightly closed in $R$ for every ideal $I \subseteq R$ [HH90, Def. 4.5];
(c) $R$ is $F$-regular if $R_p$ is weakly $F$-regular for every prime ideal $p \subseteq R$ [HH90, Def. 4.5]; and
(d) $R$ is $F$-rational if $I$ is tightly closed in $R$ for every ideal $I$ generated by a sequence of parameters in $R$ [FW89, Def. 1.10].

We note that (a) is not the usual definition of strong $F$-regularity, although it coincides with the usual definition (Definition 6(a)) for $F$-finite rings; see Theorem 8. We also note that the original definition of $F$-regularity asserted that localizations at every multiplicative set are weakly $F$-regular, but the definition using prime ideals is equivalent by [HH90, Cor. 4.15].

Next, we define $F$-singularities via purity of homomorphisms involving the Frobenius. We recall that a ring homomorphism $\varphi: R \to S$ is pure if the homomorphism

$$\varphi \otimes \text{id}: R \otimes_R M \to S \otimes_R M$$

is injective for every $R$-module $M$. To simplify notation, we fix the following:

**Notation 4.** Let $R$ be a noetherian ring of characteristic $p > 0$. For every $c \in R$ and every integer $e > 0$, we denote by $\lambda_c^e$ the composition

$$R \xrightarrow{F^e} F^e_* R \xrightarrow{F^e_* (\cdot - c)} F^e_* R.$$
**Definition 5** (*F*-singularities via purity). Let $R$ be a noetherian ring of characteristic $p > 0$. For $c \in R$, we say that $R$ is *F*-pure along $c$ if $\lambda^e_c$ is pure for some $e > 0$. Moreover, we say that

(a) $R$ is *F*-pure regular if it is *F*-pure along every $c \in R^\circ$ [HH94, Rem. 5.3];
(b) $R$ is *F*-pure if it is *F*-pure along $1 \in R$ [HR76, p. 121]; and
(c) $R$ is strongly *F*-rational if for every $c \in R^\circ$, there exists $e_0 > 0$ such that for all $e \geq e_0$, the homomorphism $\lambda^e_c \otimes R/I$ is injective for every ideal $I \subseteq R$ generated by a sequence of parameters in $R$ [Vél95, Def. 1.2].

The terminology *F*-pure regular is from [DS16, Def. 6.1.1] to distinguish it from the definition using tight closure (Definition 3(a)). *F*-pure regular rings are also called very strongly *F*-regular [Has10, Def. 3.4].

We note that *F*-purity is a local condition [DS16, Lem. 6.1.4(e)]. Strong *F*-regularity is a local condition [Has10, Lem. 3.6], and while it is equivalent to *F*-pure regularity in the local case [Has10, Lem. 3.6], *F*-pure regularity is not known to be a local condition [DS16, Rem. 6.3.3].

Next, we define *F*-singularities via splitting of homomorphisms involving the Frobenius. We use the same notation as for *F*-singularities defined using purity (Notation 4).

**Definition 6** (*F*-singularities via splitting). Let $R$ be a noetherian ring of characteristic $p > 0$. For $c \in R$, we say that $R$ is *F*-split along $c$ if $\lambda^e_c$ splits as an $R$-module homomorphism for some $e > 0$. Moreover, we say that

(a) $R$ is *split* *F*-regular if it is *F*-split along every $c \in R^\circ$ [HH94, Def. 5.1]; and
(b) $R$ is *F*-split if it is *F*-split along $1 \in R$ [MR85, Def. 2].

The terminology *split* *F*-regular is from [DS16, Def. 6.6.1]. *Split* *F*-regularity is usually known as strong *F*-regularity in the literature.

Finally, we define *F*-injective singularities.

**Definition 7** [Fed83, Def. on p. 473]. A noetherian local ring $(R, m)$ of characteristic $p > 0$ is *F*-injective if the $R$-module homomorphism

$$H^i_{m}(F): H^i_{m}(R) \longrightarrow H^i_{m}(F^i_{*}R)$$

induced by Frobenius is injective for all $i$. An arbitrary noetherian ring $R$ of characteristic $p > 0$ is *F*-injective if $R_m$ is *F*-injective for every maximal ideal $m \subseteq R$.

The relationship between these classes of singularities can be summarized as follows:

**Theorem 8.** Let $R$ be a noetherian ring of characteristic $p > 0$. We have the following diagram of implications of properties of $R$:
where “C–M” (resp. “Gor.”) is an abbreviation for Cohen–Macaulay (resp. Gorenstein).

Proof. We first list the implications that are easy or appear in the literature.

<table>
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<td>$F$-pure + weakly normal domain $\Rightarrow$ weakly normal</td>
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We now show the remaining implications, for which we could not find a reference.

$F$-pure $\Rightarrow$ weakly normal. We adapt the proof of [Sch09, Thm. 4.7]. It suffices to show that if $R$ is $F$-pure, then $R_p$ is weakly normal for every prime ideal $p \subseteq R$ by [Man80, Cor. IV.4]. Suppose not, and choose a prime ideal $p \subseteq R$ of minimal height such that $R_p$ is not weakly normal. The local ring $R_p$ is $F$-pure by [DS16, Lem. 6.1.4(e)] hence $F$-injective and reduced. Moreover, the punctured spectrum $\text{Spec}(R_p) \setminus \{p R_p\}$ is weakly normal by the minimality of $p$, hence [Sch09, Lem. 4.6] implies $R_p$ is weakly normal, a contradiction.

Weakly $F$-regular + Gorenstein away from isolated points $\Rightarrow$ Cohen–Macaulay $\Rightarrow$ strongly $F$-regular. Let $R$ be the weakly $F$-regular ring that is Cohen–Macaulay, and also Gorenstein away from isolated points. Then, the localization $R_m$ is weakly $F$-regular for every maximal ideal $m \subseteq R$ by [HH90, Cor. 4.15], and to show that $R$ is strongly $F$-regular, it suffices to show that $0$ is tightly closed in

$$E_m := E_{R_m} (R/m)$$
for every maximal ideal \( m \subseteq R \) [Has10, Lem. 3.6]. Since \( R_m \) is weakly \( R \)-regular, every submodule of a finitely generated module is tightly closed [HH90, Prop. 8.7], hence the finitist tight closure \( 0^f_E \) as defined in [HH90, Def. 8.19] is zero. Finally, since \( 0^f_E = 0^a_E \) under the assumptions on \( R \) [LS01, Thm. 8.8], we see that 0 is tightly closed in \( E_m \), hence \( R \) is strongly \( F \)-regular.

**Weakly \( F \)-regular + \( N \)-graded ⇒ split \( F \)-regular.** We adapt the proof of [LS99, Cor. 4.4]. Let \( R \) be the \( N \)-graded ring with irrelevant ideal \( m \); note that by assumption in [LS99, §3], the ring \( R \) is finitely generated over a field \( R_0 = k \) of characteristic \( p > 0 \). The localization \( R_m \) of \( R \) is weakly \( F \)-regular by [HH90, Cor. 4.15]. Now let \( L \) be the perfect closure of \( k \), and let \( m' \) be the expansion of \( m \) in \( R \otimes_k L \); since \( R \) is graded, \( m' \) is the irrelevant ideal in \( R \otimes_k L \). The ring homomorphism

\[
R_m \longrightarrow R_m \otimes_k L \cong (R \otimes_k L)_{m'}
\]

is purely inseparable and \( m \) expands to \( m' \), hence \( (R \otimes_k L)_{m'} \) is weakly \( F \)-regular by [HH94, Thm. 6.17(b)]. By the proof of [LS99, Cor. 4.3], \( R \otimes_k L \) is strongly \( F \)-regular. Finally, \( R \) is a direct summand of \( R \otimes_k L \) as an \( R \)-module, hence \( R \) is strongly \( F \)-regular as well [HH94, Thm. 5.5(e)].

**\( F \)-rational + \( F \)-finite ⇒ strongly \( F \)-rational.** The hypotheses of [Vel95, Thm. 1.12] are satisfied when the ring \( R \) is \( F \)-finite since an \( F \)-finite ring is excellent and is isomorphic to a quotient of a regular ring of finite Krull dimension by [Gab04, Rem. 13.6].

**Remark 9.** The condition that \( R \) is the image of a Cohen–Macaulay ring is not too restrictive in practice. For instance, it suffices for \( R \) to have a dualizing complex [Kaw02, Cor. 1.4], which in turn is implied by \( F \)-finiteness [Gab04, Rem. 13.6].

**Remark 10.** In the implication Weakly \( F \)-regular + Gorenstein away from isolated points + Cohen–Macaulay ⇒ strongly \( F \)-regular, MacCrimmon [Mac96, Thm. 3.3.2] showed that for \( F \)-finite rings, the Gorenstein condition can be weakened to being \( \mathbf{Q} \)-Gorenstein away from isolated points. The implication weakly \( F \)-regular + \( F \)-finite ⇒ split \( F \)-regular is a famous open problem, and is known in dimensions at most three by [Wil95, §4]. See also [Abe02] for other situations in which this implication is known and for a proof of MacCrimmon’s theorem.

**Remark 11.** The stated cases for the implication “\( F \)-rational ⇒ \( F \)-injective” follow by reducing to the local case, which is proved in [QS17, Thm. 3.7]. Thus, the implication “\( F \)-rational ⇒ \( F \)-injective” holds under different hypotheses by using [AHH93, Thm. 5.21], which shows that \( F \)-rationality localizes under various assumptions. In particular, by [AHH93, Thm. 5.21(b)], it suffices to assume that \( R \) has a weak test element and that \( R/p \) is of acceptable type (in the sense of [AHH93, p. 87]) for every minimal prime ideal \( p \subseteq R \).

**Acknowledgments.** I would like to thank Rankeya Datta for pointing out the implication “weakly \( F \)-regular + \( N \)-graded ⇒ split \( F \)-regular,” and for finding a correct reference for the implication “\( F \)-rational + local ⇒ \( F \)-injective.”

**References**


F-SINGULARITIES FOR NON-F-FINITE RINGS


