Competitive and Cooperative Inventory Policies in a Two-Stage Supply-Chain

(G. P. Cachon and P. H. Zipkin)

Presented by

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IOE 641, Supply Chain Management, Winter 2009
University of Michigan, Ann Arbor
Outline

- Introduction
- Model
- Centralized optimization problem
- Inventory games
- Nash equilibrium outcomes vs. centralized solution
- Optimal linear contracts
- Conclusion
- Future scope
Overview

- Two stage serial supply chain
- Stationary stochastic demand
- Fixed transportation time
- Single product
- Inventory holding costs at each stage
- Consumer backorder penalty at each stage
Motivation

- Retailers
  - Kroger
  - JCPenny
  - Best Buy

- Suppliers
  - Kellogg
  - Nike
  - Apple
Contribution

- Models competitive behavior of agents
  - Game theoretic analysis
- Each agent has equal position in the game
  - Analysis of Nash equilibria
- Study of two different games
  - Echelon inventory tracking
  - Local inventory tracking
- Design of linear transfer payments that help minimize system cost at Nash equilibrium
The Model

Time

- Time is slotted
The Model

- Time is slotted
- Flow of product: Source → Supplier → Retailer
The Model

Time is slotted

Flow of product: Source $\rightarrow$ Supplier $\rightarrow$ Retailer

Supplier and retailer submit the orders
The Model

- Time is slotted
- Flow of product: Source → Supplier → Retailer
- Supplier and retailer submit the orders
- Shipments are immediately released
- Lead time: Source to Supplier ($L_2$), Supplier to Retailer ($L_1$)
The Model

Demand:
- $D^\tau$ random total demand over $\tau$ periods
- Stationary distribution: density $\phi^\tau$, distribution $\Phi^\tau$
- Demand is a continuous random variable
- Positive demand occurs in each period
The Model

Inventory levels of interest:

- In transit inventory: $IT_{it}$, Supplier ($i = 2$), Retailer ($i = 1$)
- Echelon inventory level: $IL_{it}$, all inventory at stage $i$ or lower in the system minus consumer backorders
- Local inventory level: $IL_{it}$, inventory at stage $i$ minus backorders at stage $i$
The Model

Order quantity

Time

$\text{Source}$

$\text{Supplier}$

$\text{Retailer}$

$\text{Inventory levels of interest:}$

- **Echelon inventory position:** $IP_{it} = IL_{it} + IT_{it}$
- **Local inventory position:** $IP_{it} = IL_{it} + IT_{it}$
The Model

Order quantity

Time

Source

Supplier

Retailer

Order quantity

Time

$t - L_2$ $L_2$ $t - 1$ $t$ $t + 1$ $L_1$ $t + L_1 - 1$ $t + L_1$
The Model

Order quantity

Source

Supplier

Retailer

Time

Order quantity

The Model
The Model

\[
\text{Source} \quad \text{Supplier} \quad \text{Retailer}
\]

\[
\begin{align*}
\text{Time} & \quad t - L_2 & \quad t - 1 & \quad t & \quad t + 1 & \quad t + L_2 - 1 & \quad t + L_2 \\
\text{Order quantity} & \quad L_2 & \quad L_1
\end{align*}
\]
The Model

Holding costs:

- Supplier: $h_2$ per period for each unit in its stock or en route to the retailer
- Retailer: $h_1 + h_2$ per period for each unit in its stock
- Assumption: $h_2 > 0$, $h_1 \geq 0$
The Model

\[ L_t + 1 - L_t \]

Order quantity

Time

- \( t - L_2 \)
- \( L_2 \)
- \( t - 1 \)
- \( t \)
- \( t + 1 \)
- \( t + L_1 - 1 \)
- \( t + L_1 \)

- Source
- Supplier
- Retailer
The Model

\[ \text{Order quantity} \]

\[ \begin{align*}
\text{Time} & : t - L_2 & t - 1 & t & t + 1 & t + L_1 - 1 & t + L_1 \\
\text{Source} & & & & & & \\
\text{Supplier} & & & & & & \\
\text{Retailer} & & & & & & 
\end{align*} \]

\[ L_2 \quad L_1 \]
The Model

Time

Order quantity

Source

Supplier

Retailer

$t - L_2$

$L_2$

$t - 1$

$t$

$t + 1$

$t + L_1 - 1$

$t + L_1$
The Model

Backorder costs:
- System backorder cost: $p$ per unit backorder
- Supplier: $\alpha p$
- Retailer: $(1-\alpha)p$
- Assumption: $0 \leq \alpha \leq 1$
Cost Functions

Retailer:

- Cost in period \( t \): \( \hat{G}_1(IL_{1t} - D^1) \)
  \[
  \hat{G}_1(y) = (h_1 + h_2)[y]^+ + \alpha p[y]^-
  \]

- Expected cost in period \( t + L_1 \): \( G_1(IP_{1t}) \)
  \[
  G_1(y) = E[\hat{G}_1(y - D^{L_1+1})]
  \]

- \( IP_{1t} \) depends on supplier’s order and demand up to time \( t \)
  - If \( s_2 - D^L_2 \geq s_1 \), \( IP_{1t} = s_1 \)
  - If \( s_2 - D^L_2 < s_1 \), \( IP_{1t} = s_2 - D^L_2 \).

- Total expected cost
  \[
  H_1(s_1, s_2) = E[G_1(\min\{s_2 - D^L_2, s_1\})]
  \]
Cost Functions

Supplier:

- Backorder cost in period $t$: $\hat{G}_2(\text{IL}_{1,t} - D^1)$
  \[ \hat{G}_2(y) = (1 - \alpha)p[y]^- \]

- Expected backorder cost in period $t + L_1$: $G_2(IP_{1,t})$
  \[ G_2(y) = E[\hat{G}_2(y - D^{L_1+1})] \]

- Let
  \[ \hat{H}_2(s_1, x) = h_2\mu^{L_1} + h_2[x]^+ + G_2(s_1 + \min\{x, 0\}) \]

- Total expected cost
  \[ H_2(s_1, s_2) = E[\hat{H}_2(s_1, s_2 - s_1 - D^{L_2})] \]
System optimal solution

- A system optimal solution minimizes the total average cost per period.

\[(s_1^o, s_2^o) = \arg \min_{(s_1, s_2)} H_1(s_1, s_2) + H_2(s_1, s_2)\]
Echelon Inventory (EI) Game

- Players: \( i = 1,2 \)
- Strategies: \( s_i \in \sigma = [0, S], \ i = 1,2 \)
- Payoffs: \(-H_i(s_1, s_2), \ i = 1,2\)

- Pure strategy Nash equilibrium \((s_1^e, s_2^e)\)
such that,
  \[
  s_2^e \in r_2(s_1^e) \quad s_1^e \in r_1(s_2^e)
  \]
  \[
  r_1(s_2) = \{s_1 \in \sigma \mid H_1(s_1, s_2) = \min_{x \in \sigma} H_1(x, s_2)\}
  \]
  \[
  r_2(s_1) = \{s_2 \in \sigma \mid H_2(s_1, s_2) = \min_{x \in \sigma} H_2(s_1, x)\}
  \]

- Game is common knowledge
Local Inventory (LI) Game

- Players: \( i = 1,2 \)
- Strategies: \( s_i \in \sigma = [0, S], \quad i = 1,2 \)
- Payoffs: \(-H_i(\bar{s}_1, \bar{s}_2 + \bar{s}_1), \quad i = 1,2\)

- Pure strategy Nash equilibrium \((\bar{s}^l_1, \bar{s}^l_2)\)
  such that,
  \[
  \bar{s}^l_2 \in \bar{r}_2(\bar{s}^l_1) \quad \bar{s}^l_1 \in \bar{r}_1(\bar{s}^l_2)
  \]
  \[
  \bar{r}_1(\bar{s}_2) = \{\bar{s}_1 \in \sigma | H_1(\bar{s}_1, \bar{s}_2 + \bar{s}_1) = \min_{x \in \sigma} H_1(x, \bar{s}_2 + \bar{s}_1)\}
  \]
  \[
  \bar{r}_2(\bar{s}_1) = \{\bar{s}_2 \in \sigma | H_2(\bar{s}_1, \bar{s}_2 + \bar{s}_1) = \min_{x \in \sigma} H_2(\bar{s}_1, x + \bar{s}_1)\}
  \]

- Game is common knowledge
Nash equilibria

- **Theorem 4:** For $0 < \alpha < 1$, EI game has a unique Nash equilibrium.

  $$(s_1^e = s_1^a, s_2^e = r_2(s_1^a))$$

  $$\Phi^{l_{1+1}}(s_1^a) = \frac{\alpha p}{h_1 + h_2 + \alpha p}.$$

- **Theorem 8:** For $0 < \alpha < 1$, LI game has a unique Nash equilibrium.
Theorem 9: For $\alpha = 1$, EI game has the following Nash equilibria.

$$(s_1^e \in [s_2^e, S], s_2^e \in [0, s_1^a])$$

Theorem 10: For $\alpha = 1$, LI game has a unique Nash equilibrium.

$$(\bar{s}_1^l = \bar{r}_1(0), \bar{s}_2^l = 0)$$

Theorem 11: For $\alpha = 0$, both EI and LI games have unique Nash equilibrium and they are identical.

$$(s_1^e = 0, s_2^e = r_2(0))$$

$$(\bar{s}_1^l = s_1^e, \bar{s}_2^l = s_2^e - \bar{s}_1^l)$$
Comparing Nash equilibria

- **Theorem 12:** For $0 < \alpha < 1$, the base stock levels for both firms are higher in the LI game equilibrium than in the EI game equilibrium, i.e. $s^l_2 > s^e_2$ and $s^l_1 > s^e_1$

- **Theorem 13:** For $0 < \alpha < 1$, the supplier’s cost in the LI game equilibrium is lower than its cost in the EI game equilibrium.
Nash equilibria and Optimal Solution

- **Theorem 14:** In an EI game equilibrium, the retailer’s base stock level is lower than in the optimal solution.

- **Theorem 15 & 16:** For $\alpha \leq 1$, the supplier’s base stock level in both the LI and the EI equilibria is lower than in the system optimal solution.

- **Theorem 17:** For $\alpha < 1$, the system optimal solution is not a Nash equilibrium in either game.

- **Theorem 18:** For $\alpha = 1$, the system optimal solution is a Nash equilibrium in the LI game only when

  \[
  \Phi^{L_2+L_1+1}(s^o_1) = \frac{p}{h_1 + h_2 + p}
  \]
Linear Contracts

- Period $t$ transfer payment from supplier to retailer
  \[ \nu_1 I_{1t} + \beta_2 B_{2t} + \beta_1 B_{1t} \]

- Expected transfer payment in period $t + L_1$ due to retailer inventory and backorders
  \[ T_1(y) = E[\nu_1[y - D^{L_1+1}]^+ + \beta_1[y - D^{L_1+1}]^-] \]

- Expected per period transfer payment from supplier to retailer
  \[ T(\bar{s}_1, \bar{s}_2) = E[\beta_2[\bar{s}_2 - D^{L_2}]^-] \]
System with modified costs

- Costs accounting for transfer payments

\[
H_1^c(\bar{s}_1, \bar{s}_1 + \bar{s}_2) = H_1(\bar{s}_1, \bar{s}_1 + \bar{s}_2) - T(\bar{s}_1, \bar{s}_2)
\]

\[
H_2^c(\bar{s}_1, \bar{s}_1 + \bar{s}_2) = H_2(\bar{s}_1, \bar{s}_1 + \bar{s}_2) + T(\bar{s}_1, \bar{s}_2)
\]

- **Objective:** To determine the set of contracts, \((\nu_1, \beta_2, \beta_1)\), such that \((\bar{s}_1^c, \bar{s}_2^c)\) is a Nash equilibrium for the cost functions \(H_i^c(\bar{s}_1, \bar{s}_1 + \bar{s}_2)\), where \(\bar{s}_1^c = s_1^o\), and \(\bar{s}_1 + \bar{s}_2^c = s_2^o\). 
Finding optimal linear contract

- Assuming $H_i^c$ to be strictly concave in $\bar{s}_i$, find the contracts satisfying

$$\frac{\partial H_1^c}{\partial \bar{s}_1} = 0 \quad \text{and} \quad \frac{\partial H_2^c}{\partial \bar{s}_2} = 0$$

at systems optimal

- Out of this set of contracts, select the subset of contracts that make the cost functions strictly concave.
Set of optimal linear contracts

- **Theorem 19:** When the firms choose a contract \((\nu_1, \beta_2, \beta_1)\) that satisfies the following properties,

\[
(1 - \alpha) p = \left(\frac{p}{h_1 + h_2}\right) \nu_1 - \beta_1,
\]

\[
h_2 = \left(\frac{h_2}{h_1 + h_2}\right) \nu_1 + \left(\frac{1 - \gamma_2}{\gamma_2}\right) \beta_2.
\]

\[
h_1 + h_2 > \nu_1 \geq 0
\]

\[
\beta_2 > 0
\]

\[
\alpha p > \beta_1 \geq -(1 - \alpha) p,
\]

then the optimal policy is a Nash equilibrium.
Conclusion

When both players care about consumer backorders, there is a unique Nash equilibrium in EI game as well as LI game, and these equilibria differ.

The Nash equilibrium of such EI and LI game does not provide optimal solution of supply chain.

Competition lowers the supply chain inventory relative to the optimal solution.

Appropriate linear contracts can help achieving optimal supply chain solution at some Nash equilibrium.
Future Scope

- Multi product supply chains where demands of different products are correlated and are stationary, can be studied similarly by considering joint distribution of demands of these products.

- Other contracts can be investigated which ensure that all Nash equilibria provide optimal supply chain solution.

- The work can be extended to incorporate processing times of orders.
Thank you!