Transformation of generic differentiable functions to convex functions:

Necessary and sufficient conditions

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IOE 611, Winter 2006

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Outline

Motivation
A wireless communication problem
How to deal with non-convex problems ?
Convexification of non-convex functions
Necessary conditions
Sufficient conditions
Example
Checking the necessary conditions for the wireless communication problem
Conclusions

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A wireless communication problem

Power allocation for a mesh network

A wireless communication problem

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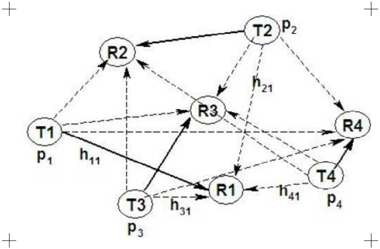

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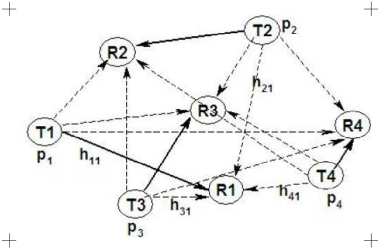

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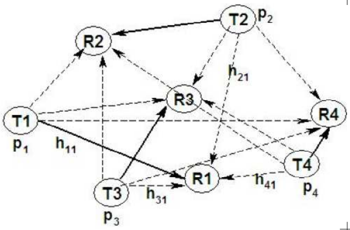

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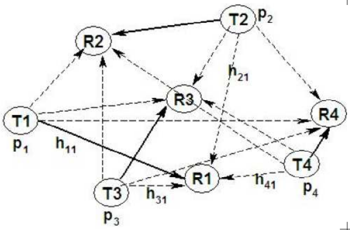

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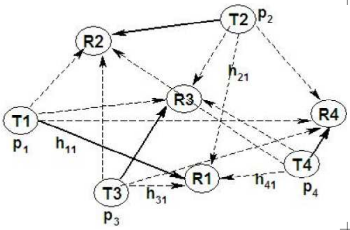

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- Objective: $\max_{p_1, p_2, \ldots, p_N} \quad \sum_{j=1}^{N} U_j(\gamma_j)$
- Problem: Objective is not concave in $(p_1, p_2, \ldots, p_N)$

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Questions

- What can we do with unfamiliar non-convex problems?
- Take a course on "Non-convex optimization"? IOE ???
- Does there exist a convex equivalent of every non-convex problem?
- Are there any conditions that characterize the class of functions which can be convexified?

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- Necessary conditions
- Sufficient conditions

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- $\phi(x) : \mathbf{R}^n \rightarrow \mathbf{R}$, twice continuously differentiable, **dom** $\phi = D$, τ sublevel set L_τ, $\alpha = \inf \phi(x)$, $\beta = \sup \phi(x)$.

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  Note: Monotonicity of F implies that ϕ and f have same minimizers.
- The derivatives of $f(x)$ can be written as:

$$\nabla f(x) = F'(\phi(x)) \, \nabla \phi(x) \tag{1}$$

$$\nabla^2 f(x) = F''(\phi(x)) \, \nabla \phi(x) \, \nabla \phi(x)^T + F'(\phi(x)) \, \nabla^2 \phi(x) \tag{2}$$

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  - the quadratic form $y^T \nabla^2 \phi(x) y$ restricted to the hyperplane $\nabla\phi(x)^T y = 0$ be positive semidefinite,
  - if $r - 1$ is its rank, the rank of the same form without the restriction be atmost r.

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- $A \in \mathbf{S}^n_+$ iff all its eigen values are non-negative.
- For each fixed $x \notin L_\alpha$ the characteristic determinant of A is,

$$\begin{aligned}
C_A(\lambda) &= \left|\nabla^2\phi + \sigma\nabla\phi\,\nabla\phi^T - \lambda I\right| \\
&= \begin{vmatrix} \nabla^2\phi - \lambda I + \sigma\nabla\phi\,\nabla\phi^T & \nabla\phi \\ 0 & 1 \end{vmatrix} \\
&= \begin{vmatrix} \nabla^2\phi - \lambda I & \nabla\phi \\ -\sigma\,\nabla\phi^T & 1 \end{vmatrix} \\
&= \left|\nabla^2\phi - \lambda I\right| - \sigma \begin{vmatrix} \nabla^2\phi - \lambda I & \nabla\phi \\ \nabla\phi^T & 0 \end{vmatrix} \qquad (6)
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- The characteristic polynomial $C_A(\lambda)$ can be written as (with $T_0 = 1$),

$$C_A(\lambda) = (-1)^n T_0 \lambda^n + (-1)^{n-1} T_1 \lambda^{n-1} + \ldots - T_{n-1}\lambda + T_n \tag{7}$$

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- Lemma 1:
  Let $Q_{P*}(y)$ be the quadratic form derived by restricting $Q_P(y)$ to the hyperplane $\nabla \phi^T y = 0$, and let its characteristic polynomial be $C_{P*}(\lambda)$. Then,

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- Proof:
The eigen values of P^* are the stationary values of $Q_P(y)$ subject to the constraints $\nabla\phi^T y = 0$ and $y^T y = 1$.
KKT conditions: For $z, \lambda \in \mathbf{R}$ and $y \in \mathbf{R}^n$,

$$L(y, z, \lambda) = y^T \nabla^2 \phi\, y + 2z \nabla\phi^T y - \lambda(y^T y - 1)$$

$$\Rightarrow 2[\nabla^2 \phi\, y + z\nabla\phi - \lambda y] = 0 \quad (10)$$

$$\nabla\phi^T y = 0 \quad (11)$$

$$y^T y = 1 \quad (12)$$

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- Obtaining the coefficient of λ^{n-1} for normalization,

$$\begin{vmatrix} -I & \nabla\phi \\ \nabla\phi^T & 0 \end{vmatrix} = (-1)^n \nabla\phi^T\nabla\phi$$

$$\Rightarrow C_{P^*}(\lambda) = -\frac{1}{k^2}\begin{vmatrix} \nabla^2\phi - \lambda I & \nabla\phi \\ \nabla\phi^T & 0 \end{vmatrix} \quad \text{where } k^2 = \nabla\phi^T\nabla\phi \qquad (14)$$

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- Further, let

$$C_{P^*}(\lambda) = (-1)^{n-1} S_0^* \lambda^{n-1} + (-1)^{n-2} S_1^* \lambda^{n-2} + \ldots - S_{n-2}^* \lambda + S_{n-1} \qquad (16)$$

$$\text{Then} \quad T_\rho = S_\rho + \sigma k^2 S_{\rho-1}^*, \quad \rho = 1, 2, \ldots, n \qquad (17)$$

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- $A \in S^n_+$ iff all the roots of $C_A(\lambda)$ are non-negative,

$$\Leftrightarrow T_\rho \geq 0, \qquad \rho = 1, 2, \ldots, n \tag{18}$$

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- For $y : \nabla \phi^T y = 0$, $Q_{P^*}(y) = Q_A(y), \Rightarrow Q_{P^*}(y) \geq 0$

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- If the rank of P^* is $r - 1$,

$$\mu^*_1 > 0, \ldots, \mu^*_{r-1} > 0, \mu^*_r = \ldots = \mu^*_{n-1} = 0$$
$$\Rightarrow \mu_1 > 0, \ldots, \mu_{r-1} > 0, \mu_r \geq 0, \mu_{r+1} = \ldots = \mu_{n-1} = 0, \mu_n \leq 0 \tag{20}$$

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- For $x \notin L_\alpha$, since $S_\rho = S^*_{\rho-1} = 0$ for $\rho > r$, $\Rightarrow T_\rho = S_\rho + \sigma k^2 S^*_{\rho-1} = 0$ for $\rho > r$. Therefore the requirement $T_\rho \geq 0$ is equivalent to,

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- If ρ_0 is the maximizer of (22), define

$$\begin{aligned} \overline{T}_\rho &:= S_\rho + \overline{\sigma} k^2 S^*_{\rho-1} \begin{cases} \geq 0, & \rho = 1, 2, \ldots, \rho_0 - 1 \\ = 0, & \rho \geq \rho_0 \end{cases} \\ \Rightarrow \overline{\sigma} &= \frac{-S_r}{k^2 S^*_{r-1}} \quad \text{since } r \geq \rho_0 \end{aligned}$$

Proof of (C), (cont')

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- The condition (22) must hold for all $x \notin L_\alpha$. Therefore,

$$\frac{F''(\tau)}{F'(\tau)} = \sigma(\phi(x)=\tau) \geq \sup_{x:\phi(x)=\tau} \overline{\sigma}(x) = \sup_{x:\phi(x)=\tau} \left(\frac{-S_r(x)}{k(x)^2 S^*_{r-1}(x)} \right) \tag{23}$$

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Proof:

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$$y^T\, \nabla^2 f(x)\, y = F''(\phi(x))\, y^T (\nabla\phi(x)\, \nabla\phi(x)^T) y + F'(\phi(x))\, y^T\, \nabla^2\phi(x)\, y \geq 0 \quad (24)$$

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- Because of (C),

$$\begin{aligned} Q_A(y) &= y^T\, \nabla^2\phi(x)\, y + \frac{F''(\phi(x))}{F'(\phi(x))}\, y^T (\nabla\phi(x)\, \nabla\phi(x)^T) y && (25) \\ &\geq y^T\, \nabla^2\phi(x)\, y - \frac{S_r(x)}{k^2 S^*_{r-1}(x)}\, y^T (\nabla\phi(x)\, \nabla\phi(x)^T) y && (26) \\ &= y^T\, A' y && (27) \end{aligned}$$

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- $\Rightarrow A'$, and hence A is positive semidefinite.

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$$\phi(p) = \gamma_1(p) = \frac{p_1 h_{11}}{N_0 + \frac{1}{B} p_2 h_{21}} \quad (30)$$

$$\Rightarrow \nabla\phi(p) = \begin{bmatrix} \frac{h_{11}}{N_0 + \frac{1}{B} p_2 h_{21}} \\ \frac{-p_1 h_{11}}{B(N_0 + \frac{1}{B} p_2 h_{21})^2} h_{21} \end{bmatrix} \quad (31)$$

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- To check (B),

$$\nabla\phi(p)^T y = 0$$

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- Necessary condition B is not satisfied.

$\Rightarrow$ There does not exist any strictly increasing function $U_1(\tau)$ that would transform the current function to a convex function :(

A numerical example

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- Power: $p_1 = 1.5, p_2 = 0.9$, Noise: $N_0 = 0.001$, Bandwidth: $B = 10000$
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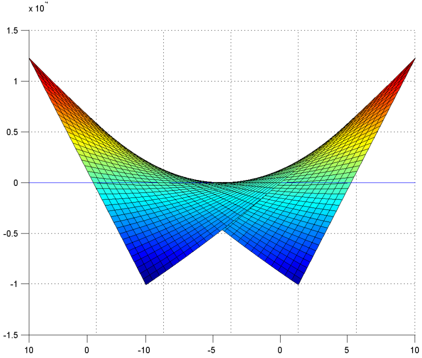

Numerical example results

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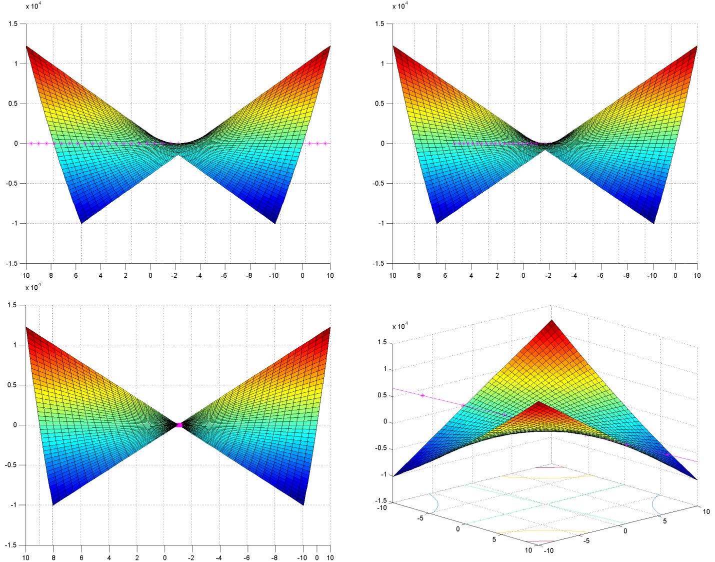

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  - First two conditions are requirements that a given function must satisfy in order to have a transformation that would make it convex.
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- Not all functions (problems) have a convex equivalent! (Shown by the example).

- If a function given to you satisfies the first two conditions, try to derive a transforming function using condition 3. You will obtain a convex problem.
  Otherwise don't waste your time!
  Look for other methods.

Acknowledgements

- Prof. Demosthenis Teneketzis
- Prof. Marina Epelman

References

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Questions?